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BOSTON UNIVERSITY

GRADUATE SCHOOL

Thesis

Solutions of Rational Integral Equations

for Complex Roots

by

Earl George Boyd

(A.B., University of Maine, 1920)

submitted in partial fulfilment of the
requirements for the degree of

Master of Arts

~~1935~~

1936

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GRADUATE SCHOOL

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Karl George Hovd

(A.B., University of Maine, 1950)

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INTRODUCTION

The purpose of this thesis is to study several methods for the complete solution of a rational integral equation when complex roots are involved, and to determine if possible the best method or methods for their solution. The only special case considered will be De Moivre's Quintic and equations of other degrees which may be solved in like manner.

I believe more attention will be given to this particular subject in the future by text book writers, since the complex unit is being used today in the practical sciences and without doubt will be more extensively used in this field in the future.

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The purpose of this thesis is to study several methods for the complete solution of a rational integral equation when complex roots are involved, and to determine if possible the best method or methods for their solution. The only special case considered will be De Moivre's formula and equations of other degrees which may be solved in like manner.

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SYNOPSIS of THEOREMS for FINDING REAL ROOTS of a RATIONAL INTEGRAL EQUATION.

An algebraic equation of the N^{th} degree may be written with all its terms transposed to the first member, thus:

$$X^n + A_1 X^{n-1} + A_2 X^{n-2} + \dots + A_{n-1} X + A_n = 0;$$

and if all the coefficients and the absolute term are real numbers, this is commonly called a numerical equation.

The first member may be denoted for brevity by $F(X)$ and the equation itself by $F(X) = 0$.

The following principles of the theory of algebraic equations with real coefficients, deduced in text-books on algebra, are here recapitulated for convenience of reference:

(1) If X_1 is a root of the equation, $F(X)$ is divisible by $(X-X_1)$, and conversely, if $F(X)$ is divisible by $X-X_1$, then X_1 is a root of the equation.

(2) An equation of the N^{th} degree has N roots and no more.

(3) If X_1, X_2, \dots, X_n are the roots of the equation, then the product $(X-X_1)(X-X_2)\dots(X-X_n)$ is equal to $F(X)$.

(4) The sum of the roots is equal to $-A_1$; the sum of the products of the roots, taken two in a set, is equal to $+A_2$; the sum of the products of the roots, taken three

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The following principles of the theory of algebraic equations with real coefficients, deduced in text-books on algebra, are here recapitulated for convenience of reference:

(1) If X_1 is a root of the equation, $V(X)$ is divisible by $(X - X_1)$, and conversely, if $V(X)$ is divisible by $(X - X_1)$, then X_1 is a root of the equation.

(2) An equation of the n^{th} degree has n roots and

no more.

(3) If X_1, X_2, \dots, X_n are the roots of the equation, then the product $(X - X_1)(X - X_2) \dots (X - X_n)$ is equal to $V(X)$.

(4) The sum of the roots is equal to $-A_1$; the sum of the products of the roots, taken two in a row, is equal to $\frac{1}{2} A_2$; the sum of the products of the roots, taken three

in a set, is equal to $-A_3$; and so on. The product of all the roots is equal to $-A_n$ when N is odd, and to $+A_n$ when N is even.

(5) The equation $F(X) = 0$ may be reduced to an equation lacking its second term by substituting $Y - A_1/N$ for X .

(6) If an equation has imaginary roots, they occur in pairs of the form $P \pm QI$ where I represents $\sqrt{-1}$.

(7) An equation of odd degree, has at least one real root whose sign is opposite to that of A_n .

(8) An equation of even degree, having A_n negative, has at least two real roots, one being positive and the other negative.

(9) A complete equation cannot have more positive roots than variations in the signs of its terms, nor more negative roots than permanences in signs. If all roots be real, there are as many positive roots as variations, and as many negative roots as permanences.

(10) In an incomplete equation, if an even number of terms, say $2m$, are lacking between two other terms, then it has at least $2m$ imaginary roots; if an odd number of terms, say $2m + 1$, are lacking between two other terms, then it has at least either $2m + 2$ or $2m$ imaginary roots, according as the two terms have like or unlike signs.

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(5) The equation $V(X) = 0$ may be reduced to an

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(6) If an equation has imaginary roots, they occur

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then it has at least either $2m + 2$ or $2m$ imaginary roots,

according as the two terms have like or unlike signs.

(11) Sturm's theorem gives the number of real roots, provided that they are unequal, as also the number of real roots lying between two assumed values of X .

(12) If A_r is the greatest negative coefficient, and if A_s is the greatest negative coefficient after X is changed into $-X$, then all real roots lie between the limits $A_r / 1$ and $-(A_s / 1)$.

Repeating this process as many times as is necessary to give the desired power to separate the roots, we may do this in the following way.

Let the given equation be

$$(I) \quad f(x) = A_0 x^n + A_1 x^{n-1} + A_2 x^{n-2} + \dots + A_{n-1} x + A_n = 0$$

and the roots be $R_1, R_2, R_3, \dots, R_n$. Then (I) may be written

$$(II) \quad f(x) = A_0 (x - R_1)(x - R_2)(x - R_3) \dots (x - R_n)$$

and multiplying (II) by the function

$$(-1)^n f(-x) = (-1)^n A_0 (-x - R_1)(-x - R_2)(-x - R_3) \dots (-x - R_n)$$

and we have

$$(III) \quad (-1)^n f(-x) \cdot f(x) = A_0^2 (x^2 - R_1^2)(x^2 - R_2^2)(x^2 - R_3^2) \dots (x^2 - R_n^2)$$

Let $X^2 = z$. Then (III) becomes

$$(IV) \quad F(z) = A_0^2 (z - R_1^2)(z - R_2^2)(z - R_3^2) \dots (z - R_n^2)$$

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A BRIEF OUTLINE of the ROOT SQUARING PROCESS

The principle is to derive an equation whose roots are powers of the roots of the given equation. The power being large enough so that the roots are widely separated. This is done by finding the equation whose roots are the squares of the roots of the original equation and then repeating the process with this new derived equation. Repeating this process as many times as is necessary to give the desired power to separate the roots. We may do this in the following way.

Let the given equation be

$$(I) \quad F(X) \equiv A_0 X^n + A_1 X^{n-1} + A_2 X^{n-2} + \dots + A_{n-1} X + A_n = 0$$

and the roots be $R_1, R_2, R_3, \dots, R_n$. Then (I) may be written

$$(II) \quad F(X) \equiv A_0 (X-R_1)(X-R_2)(X-R_3)\dots(X-R_n)$$

now multiplying (II) by the function

$$(-1)^n F(-X) \equiv (-1)^n A_0 (-X-R_1)(-X-R_2)(-X-R_3)\dots(-X-R_n)$$

and we have

$$(III) \quad (-1)^n F(-X) F(X) \equiv A_0^2 (X^2-R_1^2)(X^2-R_2^2)(X^2-R_3^2)\dots(X^2-R_n^2)$$

Let $X^2 = Z$. Then (III) becomes

$$(IV) \quad U(Z) \equiv A_0^2 (Z-R_1^2)(Z-R_2^2)(Z-R_3^2)\dots(Z-R_n^2)$$

A BRIEF OUTLINE OF THE ROOT SQUARING PROCESS

The principle is to derive an equation whose roots are powers of the roots of the given equation. The power being large enough so that the roots are widely separated. This is done by finding the equation whose roots are the squares of the roots of the original equation and then repeating the process with this new derived equation. Repeating this process as many times as is necessary to give the desired power to separate the roots. We may do this in the following way.

Let the given equation be

$$(I) \quad P(X) = A_0 X^n + A_1 X^{n-1} + A_2 X^{n-2} + \dots + A_{n-1} X + A_n = 0$$

and the roots be $R_1, R_2, R_3, \dots, R_n$. Then (I) may

be written

$$(II) \quad P(X) = A_0 (X - R_1) (X - R_2) (X - R_3) \dots (X - R_n)$$

now multiplying (II) by the function

$$(-1)^n P(-X) = (-1)^n A_0 (-X - R_1) (-X - R_2) (-X - R_3) \dots (-X - R_n)$$

and we have

$$(III) \quad (-1)^n P(-X) P(X) = A_0^2 (X^2 - R_1^2) (X^2 - R_2^2) (X^2 - R_3^2) \dots (X^2 - R_n^2)$$

Let $X^2 = Z$. Then (III) becomes

$$(IV) \quad U(Z) = A_0^2 (Z - R_1^2) (Z - R_2^2) (Z - R_3^2) \dots (Z - R_n^2)$$

now the roots of equation (IV) are $R_1^2, R_2^2, R_3^2, \dots, R_n^2$, and are therefore the squares of the roots of equation (I).

This multiplication may be done by a fairly easy method as follows

consider the fifth degree equation

$$(V) \quad F(X) = A_0 X^5 + A_1 X^4 + A_2 X^3 + A_3 X^2 + A_4 X + A_5$$

Then

$$(VI) \quad (-1)^5 F(-X) = A_0 X^5 - A_1 X^4 + A_2 X^3 - A_3 X^2 + A_4 X - A_5$$

By actually multiplying (V) by VI we have

$$\begin{aligned} (-1)^5 F(-X) F(X) &= A_0^2 X^{10} - A_1^2 X^8 + 2A_0 A_2 X^6 \\ &+ A_2^2 X^6 - 2A_1 A_3 X^6 + 2A_0 A_4 X^6 - A_3^2 X^4 + 2A_2 A_4 X^4 \\ &- 2A_1 A_5 X^4 + A_4^2 X^2 - 2A_3 A_5 X^2 - A_5^2 \end{aligned}$$

collecting coefficients

$$(v) \quad \begin{array}{c} A_0 X^5 + A_1 X^4 + A_2 X^3 + A_3 X^2 + A_4 X + A_5 \\ \hline A_0^2 X^{10} - A_1^2 X^8 + 2A_0 A_2 X^6 - A_2^2 X^6 - 2A_1 A_3 X^6 + 2A_0 A_4 X^6 - A_3^2 X^4 + 2A_2 A_4 X^4 - 2A_1 A_5 X^4 + A_4^2 X^2 - 2A_3 A_5 X^2 - A_5^2 \end{array}$$

If we had started with an equation of even degree and performed the multiplication in like manner we would have found that the law for the formation of the coefficient would have been the same. The coefficient may

now the roots of equation (IV) are $H_1, H_2, H_3, \dots, H_n$, and
 are therefore the squares of the roots of equation (I).
 This multiplication may be done by a fairly easy

method as follows

consider the fifth degree equation

$$(V) \quad f(x) = A_0 x^5 + A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5$$

Then

$$(VI) \quad (-1)^5 f(-x) = A_0 x^5 - A_1 x^4 + A_2 x^3 - A_3 x^2 + A_4 x - A_5$$

$$- \frac{A_5}{A_0}$$

By actually multiplying (V) by VI we have

$$(-1)^5 f(x) f(-x) = A_0^2 x^{10} - A_1^2 x^9 + 2A_0 A_2 x^8 - A_1 A_3 x^7 + 2A_0 A_4 x^6 - A_1 A_5 x^5 + A_2^2 x^4 - 2A_1 A_2 A_3 x^3 + A_2^2 A_4 x^2 - 2A_1 A_3 A_4 x + A_3^2 x - A_4^2$$

$$+ A_5^2 - 2A_0 A_1 A_2 x^8 - 2A_0 A_1 A_3 x^7 - 2A_0 A_1 A_4 x^6 - 2A_0 A_2 A_3 x^5 - 2A_0 A_2 A_4 x^4 - 2A_0 A_3 A_4 x^3 - 2A_1 A_2^2 x^2 - 2A_1 A_3 A_4 x - A_2^2 A_5 - A_3^2 A_4 - A_4^2 A_5$$

$$- 2A_0 A_1 A_2 x^8 - 2A_0 A_1 A_3 x^7 - 2A_0 A_1 A_4 x^6 - 2A_0 A_2 A_3 x^5 - 2A_0 A_2 A_4 x^4 - 2A_0 A_3 A_4 x^3 - 2A_1 A_2^2 x^2 - 2A_1 A_3 A_4 x - A_2^2 A_5 - A_3^2 A_4 - A_4^2 A_5$$

collecting coefficients

$$(V) \quad \begin{array}{c|c|c|c|c|c} A_0 x^{10} & -A_1 x^9 & 2A_0 A_2 x^8 & -A_1 A_3 x^7 & 2A_0 A_4 x^6 & -A_1 A_5 x^5 \\ \hline -A_1 x^9 & A_1^2 x^8 & -2A_0 A_1 A_2 x^7 & A_1^2 A_3 x^6 & -2A_0 A_1 A_4 x^5 & A_1^2 A_5 x^4 \\ \hline 2A_0 A_2 x^8 & -2A_0 A_1 A_2 x^7 & 2A_0^2 A_2^2 x^6 & -2A_0 A_1 A_2 A_3 x^5 & 2A_0^2 A_2 A_4 x^4 & -2A_0 A_1 A_2 A_4 x^3 \\ \hline -A_1 A_3 x^7 & A_1^2 A_3 x^6 & -2A_0 A_1 A_2 A_3 x^5 & A_1^2 A_3^2 x^4 & -2A_0 A_1 A_3 A_4 x^3 & A_1^2 A_3 A_5 x^2 \\ \hline 2A_0 A_4 x^6 & -2A_0 A_1 A_4 x^5 & 2A_0^2 A_2 A_4 x^4 & -2A_0 A_1 A_2 A_4 x^3 & 2A_0^2 A_4^2 x^2 & -2A_0 A_1 A_4 A_5 x \\ \hline -A_1 A_5 x^5 & A_1^2 A_5 x^4 & -2A_0 A_1 A_2 A_5 x^3 & A_1^2 A_5 A_3 x^2 & -2A_0 A_1 A_4 A_5 x & A_1^2 A_5^2 \end{array}$$

If we had started with an equation of even degree

and performed the multiplication in like manner we would
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 cients would have been the same. The coefficient may

therefore be written down according to the following rule:

1. The numbers in the top row are the squares of the coefficients directly above them, with alternating signs--the second, fourth, sixth, etc. squared numbers being negative.

2. The quantities directly under these squared numbers are the doubled products of the coefficients equally removed from the one directly overhead, the first being twice the product of the two coefficients adjacent to the one overhead, the second the doubled product of the next two equally removed coefficients, etc.

3. The signs of the doubled products are changed alternately in going along the rows and also in going down the columns, the sign of the first doubled product in each row not being changed.

Let us assume the equation to be

$$B_0 (X^m)^n + B_1 (X^m)^{n-1} + \dots + B_{n-1} (X^m) + B_n = 0$$

after the process has been repeated enough times to widely separate the roots.

Using the following relationships between coefficients and roots

$$\frac{B_1}{B_0} = -(R_1^m + R_2^m + R_3^m + \dots + R_n^m)$$

therefore be written down according to the following rule:

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$$x^m + p_1 x^{m-1} + p_2 x^{m-2} + \dots + p_{m-1} x + p_m = 0$$

After the process has been repeated enough times to widely separate the roots.

Using the following relationships between coeffi-

cients and roots

$$\frac{p_1}{p_m} = -(r_1 + r_2 + r_3 + \dots + r_m)$$

$$\frac{B_2}{B_0} = R_1^m R_2^m / R_1^m R_3^m / \dots + R_1^m R_n^m / R_2^m R_3^m / R_2^m R_4^m$$

$$/ \dots R_2^m R_n^m / \dots + R_{n-1}^m R_n^m$$

$$\frac{B_3}{B_0} = -(R_1^m R_2^m R_3^m / R_1^m R_2^m R_4^m / \dots$$

$$\frac{B_n}{B_0} = (-1)^n R_1^m R_2^m \dots R_n^m$$

Where $R_1 > R_2 > R_3 > \dots > R_n$

Now if the roots are widely separated the first terms in the right hand members of these equations predominate. That is the others are negligible in comparison--assuming the M^{th} powers are such as to give us the required decimal accuracy.

then

$$\frac{B_1}{B_0} = -R_1^m \text{ (approximately)}$$

$$\frac{B_2}{B_0} = R_1^m R_2^m \quad "$$

$$\frac{B_3}{B_0} = -R_1^m R_2^m R_3^m \quad "$$

$$\frac{B_n}{B_0} = (-1)^n R_1^m R_2^m \dots R_n^m$$

Dividing each equation after the first by the preceding equation, we have

$$\frac{B_2}{B_1} = -R_2^m, \quad \frac{B_3}{B_2} = -R_3^m, \quad \frac{B_n}{B_{n-1}} = -R_n^m.$$

$$\frac{E_2}{E_0} = R_1^m R_2^m \dots R_n^m \dots$$

$$\frac{E_2}{E_0} = R_1^m R_2^m \dots R_n^m \dots$$

$$\frac{E_2}{E_0} = R_1^m R_2^m \dots R_n^m \dots$$

$$\frac{E_2}{E_0} = R_1^m R_2^m \dots R_n^m \dots$$

Where $R_1 > R_2 > R_3 > \dots > R_n$

Now, if the roots are widely separated the first

terms in the right hand members of these equations pre-
dominate. That is the others are negligible in compari-
son--assuming the R_1^m powers are such as to give us the
required decimal accuracy.

then

$$\frac{E_1}{E_0} = -R_1^m \text{ (approximately)}$$

$$\frac{E_2}{E_0} = R_1^m R_2^m$$

$$\frac{E_3}{E_0} = -R_1^m R_2^m R_3^m$$

$$\frac{E_4}{E_0} = (-1)^m R_1^m R_2^m R_3^m R_4^m$$

Dividing each equation after the first by the pre-

ceding equation, we have

$$\frac{E_2}{E_1} = -R_2^m, \frac{E_3}{E_2} = -R_3^m, \frac{E_4}{E_3} = -R_4^m, \dots$$

Now to find the roots of the equation all we need to do is solve the binomial equations.

$$B_1 \neq B_0 R_1^m = 0$$

$$B_2 \neq B_1 R_2^m = 0$$

$$B_3 \neq B_2 R_3^m = 0$$

$$\begin{array}{c} \text{-----} \\ B_n \neq B_{n-1} X_n^m = 0 \end{array}$$

The solution may be done conveniently by logarithms.

When the roots are all real. After the first two or three transformations the signs of the coefficients of the terms in the transformed equations alternate in sign and remain with the same sign for all the remaining transformations. If the roots are either positive or negative when they are squared they all become positive and the equation with all positive roots alternate in sign.

Also the doubled products disappear for the required number of decimals. As the transformations are made the coefficients become very large and it is only necessary to keep as many digits accurately as we desire accuracy in our roots. The discussion for the disappearance of the doubled products may be read on page 12. In this discussion it will be shown that if the doubled products do not disappear for a particular row that it will be due to the presence of a pair of complex roots.

Now to find the roots of the equation all we need to do

is solve the binomial equations.

$$x^2 + 2bx + c = 0$$

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The solution may be found conveniently by logarithms.

When the roots are all real, after the first two or

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same with the same sign for all the remaining transformations.

If the roots are either positive or negative when they are

equated they all become positive and the equation with all

positive roots alternate in sign.

Also the doublet product is sufficient for the required num-

ber of solutions. As the transformations are made the coeffi-

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many figures necessarily as we desire accuracy in our roots.

The discussion for the discussion of the doublet remains

may be used in case 12. In this discussion it will be shown

that if the doublet remains do not disappear for a particular

case that it will be due to the presence of a pair of

complex roots.

THE ROOT SQUARING METHOD WHERE ONE PAIR of COMPLEX ROOTS OCCUR

When the equation has a pair of complex roots it cannot be expressed as a product of linear factors with real coefficients. It may be expressed as a product of linear equations and a single quadratic, the quadratic having the pair of complex roots for its roots. The root squaring method therefore breaks it up into linear factors and a quadratic factor.

The complex root may be noted in two ways:

* (1) the doubled products do not disappear from one column and (See page 12-2nd term of quadratic fragment)

(2) the signs in this particular column fluctuate.

(See note on page 8)

$$(a \neq b, a \neq 0)$$

Assume we have $x_1, a \neq b i, a - b i, x_2$ and x_3 for the roots of an equation with the order of magnitude being $|x_1| > |r| > |x_2| > |x_3|$ where "r" is the modulus of the complex numbers which are roots of the equation that is

$$a \neq b i = r (\cos \theta \neq i \sin \theta).$$

$$\text{now } e^{i\theta} = \cos \theta \neq i \sin \theta$$

$$r e^{i\theta} = r (\cos \theta \neq i \sin \theta)$$

* The Column is dependent on the magnitude of the modulus of the complex roots compared to the real roots.

THE ROOT SQUARING METHOD WHEN ONE PAIR OF COMPLEX

ROOTS OCCUR

When the equation has a pair of complex roots it cannot be expressed as a product of linear factors with real coefficients. It may be expressed as a product of linear equations and a single quadratic, the quadratic having the pair of complex roots for its roots. The root squaring method therefore breaks it up into linear factors and a quadratic factor.

The complex root may be noted in two ways:

- (1) the doubled products do not disappear from one column and (see page 12-2nd term of quadratic fragment)
- (2) the signs in this particular column fluctuate.

(See note on page 11)
($a \neq b, a \neq 0$)

Assume we have $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$ for

the roots of an equation with the order of magnitude being $|x_1| > |x_2| > |x_3| > |x_4| > |x_5| > |x_6| > |x_7| > |x_8| > |x_9| > |x_{10}|$ where "r" is the modulus of the complex numbers which are roots of the equation that is

$$\begin{aligned} a \pm b i &= r (\cos \theta \pm i \sin \theta) \\ \text{now } e^{i\theta} &= \cos \theta + i \sin \theta \\ r e^{i\theta} &= r (\cos \theta + i \sin \theta) \end{aligned}$$

The column is dependent on the magnitude of the modulus of the complex roots compared to the real roots.

$$\text{and } e^{-i\theta} = \cos(-\theta) + i \sin(-\theta)$$

$$= \cos \theta - i \sin \theta$$

$$r e^{-i\theta} = r (\cos \theta - i \sin \theta)$$

Then the equation having these roots is

$$I \quad (x - x_1) (x - r e^{i\theta}) (x - r e^{-i\theta}) (x - x_2) (x - x_3) = 0.$$

and the equation having for its roots the m^{th} powers of the roots of equation (I) is

$$II \quad (y - x_1^m) (y - r^m e^{im\theta}) (y - r^m e^{-im\theta}) (y - x_2^m)$$

$$(y - x_3^m) = 0.$$

where $y = x^m$, Multiplying we have

$$y^5 - (x_1^m + x_2^m + x_3^m + r e^{im\theta} + r e^{-im\theta}) y^4$$

$$+ (x_1^m x_2^m + x_1^m x_3^m + \dots + x_2^m x_3^m + \dots + r^m e^{im\theta} r^m e^{-im\theta}$$

$$+ \dots) y^3$$

$$- (x_1^m x_2^m x_3^m + x_1^m x_2^m r^m e^{im\theta} + \dots + x_2^m r^m e^{im\theta} r^m e^{-im\theta}$$

$$+ \dots) y^2$$

$$+ (x_1^m x_2^m x_3^m r^m e^{im\theta} + x_1^m x_2^m x_3^m e^{-im\theta} + x_2^m x_3^m r^m e^{-im\theta}) y$$

$$- x_1^m x_2^m x_3^m r^m e^{im\theta} r^m e^{-im\theta}$$

Since $|x_1| > |r| > |x_2| > |x_3|$ the term of greatest modulus

is the predominating part of the coefficient of y^4 , x_1^m ,

$$\text{for } y^3 \text{ it is } x_1^m r^m e^{im\theta} + x_1^m r^m e^{-im\theta} =$$

$$\text{and } e^{-i\theta} = \cos(-\theta) + i \sin(-\theta)$$

$$= \cos \theta - i \sin \theta$$

$$r e^{-i\theta} = r (\cos \theta - i \sin \theta)$$

Then the equation having these roots is

$$I \quad (x - x_1)(x - r e^{i\theta})(x - x_2)(x - r e^{-i\theta}) = 0$$

and the equation having for its roots the m^{th} powers

of the roots of equation (I) is

$$II \quad (y - x_1^m)(y - r^m e^{i m \theta})(y - x_2^m)(y - r^m e^{-i m \theta}) = 0$$

$$(y - x_3^m) = 0$$

where $y = x^m$. Multiplying we have

$$y^3 - (x_1^m + x_2^m + x_3^m) y^2 + (x_1^m x_2^m + x_1^m x_3^m + x_2^m x_3^m) y - x_1^m x_2^m x_3^m = 0$$

$$+ (x_1^m x_2^m + x_1^m x_3^m + x_2^m x_3^m) y - x_1^m x_2^m x_3^m = 0$$

$$+ \dots + y^3$$

$$- (x_1^m x_2^m + x_1^m x_3^m + x_2^m x_3^m) y + x_1^m x_2^m x_3^m = 0$$

$$+ \dots + y^3$$

$$+ (x_1^m x_2^m + x_1^m x_3^m + x_2^m x_3^m) y - x_1^m x_2^m x_3^m = 0$$

$$- x_1^m x_2^m x_3^m = 0$$

since $|x_1| > |x_2| > |x_3|$ the term of greatest modulus

the predominating part of the coefficient of y^3 is

$$\text{for } x \text{ it is } x_1^m e^{i m \theta} + x_2^m e^{-i m \theta}$$

$$x_1^m r^m (e^{im\theta} + e^{-im\theta}) =$$

$$x_1^m r^m (\cos m\theta + i \sin m\theta + \cos m\theta - i \sin m\theta) =$$

$$x_1^m r^m (2 \cos m\theta)$$

$$\text{For } y^2, x_1^m (r^m e^{im\theta}) (r^m e^{-im\theta}) = x_1^m r^{2m}$$

$$\text{For } y, x_1^m r^{2m} x_2^m$$

Then we have for the transformed equation whose roots are approximately the m^{th} powers of the roots of the initial equation.

$$\text{III } y^5 - x_1^m y^4 + 2x_1^m r^m \cos(m\theta) y^3 - x_1^m r^{2m} y^2 + x_1^m r^{2m} x_2^m y - x_1^m r^{2m} x_2^m x_3^m = 0$$

Definition: The word fragment is to indicate any group of terms taking in sequence when the equation is arranged according to the descending powers of the unknown.

Now taking the following fragments from (III) and solving as equations:

$$(a) y^5 - x_1^m y^4 = 0 \quad y^4 (y - x_1^m) = 0 \quad y = x_1^m$$

$$(b) -x_1^m y^4 + 2x_1^m r^m \cos(m\theta) y^3 - x_1^m r^{2m} y^2 = 0.$$

$$-x_1^m y^2 (y^2 - 2r^m \cos(m\theta) y + r^{2m}) = 0.$$

$$y^2 - r^m [\{\cos m\theta + i \sin(m\theta)\} + \{\cos(m\theta) - i \sin m\theta\}] y - r^{2m} = 0$$

$$y^2 - r^m (e^{im\theta} + e^{-im\theta}) y - r^{2m} = 0.$$

$$(y - r^m e^{im\theta}) (y - r^m e^{-im\theta}) = 0. \quad y = r^m e^{im\theta}, \quad y = r^m e^{-im\theta}$$

$$(c) -x_1^m r^{2m} y^2 + x_1^m r^{2m} x_2^m y = 0$$

$$-x_1^m r^{2m} y (y - x_2^m) = 0 \quad y = x_2^m$$

$$(d) x_1^m r^{2m} x_2^m y - x_1^m r^{2m} x_2^m x_3^m = 0$$

$$x_1^m r^{2m} x_2^m (y - x_3^m) = 0 \quad y = x_3^m$$

$$x_1^2 (e^{i\theta} + e^{-i\theta}) =$$

$$x_1^2 (e^{i\theta} + e^{-i\theta}) =$$

$$x_1^2 (e^{i\theta} + e^{-i\theta}) =$$

$$\text{For } x_1^2, x_1^2 (e^{i\theta} + e^{-i\theta}) = x_1^2$$

$$\text{For } x_1^2, x_1^2 (e^{i\theta} + e^{-i\theta}) =$$

then we have for the transformed equation whose roots are approximately the n th powers of the roots of the original equation.

$$\text{III } y^2 - \frac{1}{2} y^4 + \frac{1}{2} x_1^2 \cos(n\theta) y - \frac{1}{2} x_1^2 = 0$$

$$y^2 - \frac{1}{2} y^4 + \frac{1}{2} x_1^2 \cos(n\theta) y - \frac{1}{2} x_1^2 = 0$$

Definition: The word transformation is to indicate any group of terms taking in accordance with the equation is changed according to the ascending powers of the unknown.

Now we have the following transformed form (III) and

solving an equation:

$$\text{(a) } y^2 - \frac{1}{2} y^4 = 0 \quad y^2 (y^2 - \frac{1}{2}) = 0 \quad y = 0 \quad y = \pm \frac{1}{\sqrt{2}}$$

$$\text{(b) } y^2 - \frac{1}{2} y^4 + \frac{1}{2} x_1^2 \cos(n\theta) y - \frac{1}{2} x_1^2 = 0$$

$$y^2 - \frac{1}{2} y^4 - \frac{1}{2} x_1^2 \cos(n\theta) y + \frac{1}{2} x_1^2 = 0$$

$$y^2 - \frac{1}{2} y^4 - \frac{1}{2} x_1^2 \cos(n\theta) y + \frac{1}{2} x_1^2 = 0$$

$$y^2 - \frac{1}{2} y^4 - \frac{1}{2} x_1^2 \cos(n\theta) y + \frac{1}{2} x_1^2 = 0$$

$$\text{(c) } y^2 - \frac{1}{2} y^4 + \frac{1}{2} x_1^2 \cos(n\theta) y - \frac{1}{2} x_1^2 = 0$$

$$y^2 - \frac{1}{2} y^4 - \frac{1}{2} x_1^2 \cos(n\theta) y + \frac{1}{2} x_1^2 = 0$$

$$\text{(d) } y^2 - \frac{1}{2} y^4 - \frac{1}{2} x_1^2 \cos(n\theta) y + \frac{1}{2} x_1^2 = 0$$

$$y^2 - \frac{1}{2} y^4 - \frac{1}{2} x_1^2 \cos(n\theta) y + \frac{1}{2} x_1^2 = 0$$

From these equations we ~~do~~ obtain the roots with which we started. **on page 9.**

Applying the root squaring process to (III) once more.

	y^5	y^4	y^3	y^2
m^0	1	$-x_1^m$	$2x_1^m r^m \cos m\theta$	$-x_1^m r^{2m}$
	1	$-x_1^{2m}$ $4x_1^m r^m \cos m\theta$	$4x_1^{2m} r^{2m} \cos^2 m\theta$ $-2x_1^{2m} r^{2m}$ $2x_1^m r^{2m} x_2^m$	$-x_1^{2m} r^{4m}$ $4x_1^{2m} r^{3m} x_2^m \cos m\theta$ $2x_1^{2m} r^{2m} x_2^m x_3^m$
$(2m)^0$	1	$-x_1^{2m}$	$4x_1^{2m} r^{2m} \cos^2 m\theta$ $-2x_1^{2m} r^{2m}$	$-x_1^{2m} r^{4m}$
y	y^0			
$x_1^m r^{2m} x_2^m$	$-x_1^m r^{2m} x_2^m x_3^m$			
$x_1^{2m} r^{4m} x_2^{2m}$ $2x_1^{2m} r^{4m} x_2^m x_3^m$	$-x_1^{2m} r^{4m} x_2^{2m} x_3^{2m}$			
$x_1^{2m} r^{4m} x_2^{2m}$	$-x_1^{2m} x_2^{4m} x_2^{2m} x_3^{2m}$			

If we divide the doubled term in each column by the squared term at the top we ^{shall} see that these **quotients** are negligible except one in the first row, column 3, These divisions will indicate the dominant terms of each coefficient.

Coefficient
 divisions will indicate the dominant terms of each
 negligible except one in the first row, column 3. These
 separated term at the top we see that these products are
 small
 If we divide the doubled term in each column by the

x_1	x_2	x_3	x_4
x_1	x_2	x_3	x_4
x_1	x_2	x_3	x_4
x_1	x_2	x_3	x_4

x_1	x_2	x_3	x_4
x_1	x_2	x_3	x_4
x_1	x_2	x_3	x_4
x_1	x_2	x_3	x_4

Applying the root separating process to (III) once more
 which we started on page 2.
 From these equations we can obtain the roots with

Taking the coefficient of y^3 as an illustration

$$\frac{2x_1^m r^{2m} x_2^m}{4x_1^{2m} r^{2m} \cos^2 m \cdot \theta} = \frac{x_2^m}{2x_1^m \cos^2 m \cdot \theta}$$

which is negligible since the roots are widely separated and

$$|x_1| > |r| > |x_2| > |x_3|.$$

If $m \theta$ is near 90° by repeating the process several times $m \theta$ will not be near enough 90° to affect the quotient from being negligible. $m \theta$ cannot be 90° since $a \neq b$ in the complex roots.

$$\text{Now } 2 \cos^2 \theta - 1 = \cos 2 \cdot \theta$$

then the coefficient of y^3 may be written

$$4x_1^{2m} r^{2m} \cos^2 m \cdot \theta - 2x_1^{2m} r^{2m}$$

$$2x_1^{2m} r^{2m} (2 \cos^2 m \cdot \theta - 1)$$

$$2x_1^{2m} r^{2m} \cos(2 m \cdot \theta)$$

Then the last transformed equation whose roots are of the $2m^{\text{th}}$ power are:

$$\begin{aligned} \text{(IV)} \quad y^5 - x_1^{2m} y^4 + 2x_1^{2m} r^{2m} \cos(2 m \cdot \theta) y^3 - x_1^{2m} r^{4m} y^2 \\ + x_1^{2m} r^{4m} + x_1^{2m} r^{4m} x_2^{2m} y - x_1^{2m} r^{4m} x_2^{2m} x_3^{2m} = 0 \end{aligned}$$

On comparing the coefficients of equation (IV) with those of equation (III) we see that the application of the root squaring method doubles the amplitude of the complex roots. As the amplitudes are doubled the cosines frequently change signs. This is the reason for the fluctuation of the signs in one of the coefficients when complex roots are present.

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root doubling method doubles the amplitude of the complex

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On comparing the coefficients of equation (IV) with

$$(IV) \quad y^2 - x_1^2 y + x_1^2 \cos(2 \cdot m \cdot e) y - x_1^2 \cos^2 m \cdot e = 0$$

2mth power are:

Then the last transformed equation whose roots are of the

$$2x_1^2 r^2 \cos(2 \cdot m \cdot e)$$

$$2x_1^2 r^2 (2 \cos^2 m \cdot e - 1)$$

$$4x_1^2 r^2 \cos^2 m \cdot e - 2x_1^2 r^2 \cos m \cdot e$$

then the coefficient of y^2 may be written

$$\text{Now } 2 \cos^2 e - 1 = \cos 2 \cdot e$$

complex roots.

being negligible. $m \cdot e$ cannot be 90° since a $\frac{1}{2}$ in the

$m \cdot e$ will not be near enough 90° to effect the quotient from

It $m \cdot e$ is near 90° by repeating the process several times

$$|x_1| > |r| > |x_2| > |x_3|.$$

which is negligible since the roots are widely separated and

$$\frac{4x_1^2 r^2 \cos^2 m \cdot e}{2x_1^2 r^2 \cos m \cdot e} = \frac{2x_1^2 r^2 \cos^2 m \cdot e}{2x_1^2 r^2 \cos m \cdot e}$$

Taking the coefficient of y^2 as an illustration

RELATIONS between the COEFFICIENTS of an ALGEBRAIC
EQUATION AND the RECIPROCALs of its ROOTS.

In the equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-2} x^2 + a_{n-1} x + a_n = 0.$$

Let $x = 1/y$ and clearing of fractions we have

$$a_n y^n + a_{n-1} y^{n-1} + a_{n-2} y^{n-2} + \dots + a_2 y^2 + a_1 y + a_0 = 0.$$

now from the relationship between roots and coefficients
 we have

$$\frac{a_{n-1}}{a_n} = -(y_1 + y_2 + y_3 + \dots + y_n)$$

$$\frac{a_{n-2}}{a_n} = y_1 y_2 + y_1 y_3 + y_1 y_4 + \dots + y_2 y_3 + y_2 y_4 + \dots$$

$$\frac{a_0}{a_n} = (-1)^n y_1 y_2 y_3 \dots y_n$$

and since $y = 1/x$

$$\frac{a_{n-1}}{a_n} = -(1/x_1 + 1/x_2 + 1/x_3 + \dots + 1/x_n)$$

$$\frac{a_{n-2}}{a_n} = 1/x_1 x_2 + 1/x_1 x_3 + 1/x_1 x_4 + \dots + 1/x_2 x_3 + 1/x_2 x_4 + \dots$$

RELATIONS BETWEEN THE COEFFICIENTS OF AN ALGEBRAIC
EQUATION AND THE SUMS OF ITS ROOTS.

In the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0.$$

Let $x = 1/y$ and clearing of fractions we have

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots + a_{n-1} y + a_n = 0.$$

Now from the relationship between roots and coefficients

we have

$$\frac{a_{n-1}}{a_n} = -(y_1 + y_2 + \dots + y_n)$$

$$\frac{a_{n-2}}{a_n} = y_1 y_2 + y_1 y_3 + \dots + y_1 y_n + y_2 y_3 + \dots + y_2 y_n + \dots + y_{n-2} y_{n-1} + y_{n-2} y_n$$

$$\frac{a_{n-3}}{a_n} = -(y_1 y_2 y_3 + y_1 y_2 y_4 + \dots + y_1 y_2 y_n + y_1 y_3 y_4 + \dots + y_1 y_3 y_n + \dots + y_1 y_{n-2} y_{n-1} + y_1 y_{n-2} y_n + \dots + y_2 y_{n-2} y_{n-1} + y_2 y_{n-2} y_n + \dots + y_{n-3} y_{n-2} y_{n-1} + y_{n-3} y_{n-2} y_n)$$

and since $y = 1/x$

$$\frac{a_{n-1}}{a_n} = -(1/x_1 + 1/x_2 + \dots + 1/x_n)$$

$$\frac{a_{n-2}}{a_n} = 1/x_1 x_2 + 1/x_1 x_3 + \dots + 1/x_1 x_n + 1/x_2 x_3 + \dots + 1/x_2 x_n + \dots + 1/x_{n-2} x_{n-1} + 1/x_{n-2} x_n$$

These relationships between the coefficients and reciprocals of the roots will assist us in avoiding ambiguities of signs in the computation of the complex roots.

Example: $x^4 - 6x^3 + 11x^2 + 2x - 28 = 0$

The small n and c will denote no change in sign or a change in sign.

	x^4	x^3	x^2	x^1	x^0
Eq.	1	-6	11	2	-28
	1	-36	121	-4	784
		22	n 24	c -616	n
			-56	n	
P ²	1	-1.4(10)	8.9(10)	-6.2(10) ²	7.84(10) ²
	1	-1.96(10) ²	7.921(10) ³	-3.844(10) ⁵	6.147(10) ⁵
		1.78	n -17.36	c 1.396	n
			1.568	n	
P ⁴	1	-.18(10) ²	-7.871(10) ³	-2.448(10) ⁵	6.147(10) ⁵
	1	-.032.10 ⁴	6.195.10 ⁷	-5.992.10 ¹⁰	3.779.10 ¹¹
		-1.574	n -.881	-.968	n
			.123	n	
P ⁸	1	-1.606(10) ⁴	5.437(10) ⁷	-6.960(10) ¹⁰	3.779(10) ¹¹
	1	-2.580(10) ⁸	2.956(10) ¹⁵	-4.843(10) ²¹	1.428(10) ²³
		1.087	n -2.236	c .041	n
			.001	n	
P ¹⁶	1	-1.493(10) ⁸	-721(10) ¹⁵	-4.802(10) ²¹	1.428(10) ²³

These relationships between the coefficients and roots of the roots will assist us in avoiding ambiguities of signs in the computation of the complex roots.

Example: $x^4 - 6x^3 + 11x^2 - 6x - 28 = 0$

The small n and c will denote no change in sign or a change in sign.

x^0	x^1	x^2	x^3	x^4
28	1	-6	11	-28
1	-36	121	24	784
	22	n 24	n -36	n
p_1 1	-1.4110	0.9110	-0.210	7.8410
1	-1.2210	7.9210	n -17.50	0.1710
	1.78	1.503	n	n
p_2 1	-1.1510	-7.8710	-0.4410	0.1710
1	-0.3210	0.12210	-0.92210	3.77910
	-1.274	n -1.31	n -0.88	n
p_3 1	-1.0010	0.4710	-0.9010	0.7710
1	-0.5010	0.9210	-4.3210	1.42310
	1.007	n -0.956	n 0.041	n
p_4 1	-1.42310	-7.9110	-4.90210	1.42310

	1	$-2.228(10)^{16}$.144 n	$.520(10)^{30}$ -1.434 c * n	$-2.305(10)^{43}$ * n	$2.038(10)^{46}$
P ³²	1	$-2.084(10)^{16}$	$-.914(10)^{30}$	$-2.305(10)^{43}$	$2.038(10)^{46}$
	1	$-4.342(10)^{32}$ - .018 n	$.835(10)^{60}$ -.960 c	$-5.314(10)^{86}$ * n	$4.153(10)^{92}$
P ⁶⁴	1	$-4.360(10)^{32}$	$-.125(10)^{60}$	$-5.314(10)^{86}$	$4.153(10)^{92}$

We see that another application of the root squaring method would not further separate the roots.

The given equation has now been broken up into two linear and one quadratic fragment, taking the first linear fragment and computing the root by logarithms.

$$x_1^{64} = 4.360 \cdot 10^{32} \quad (b_0 x^m \neq b_1 = 0)$$

$$64 \log x_1 = \log 4.360 + 32 \log 10$$

$$64 \log x_1 = .639486 + 32$$

$$\log x_1 = .509992$$

$$x_1 = 3.2359$$

Solving the second linear fragment

$$5.314 \cdot 10^{86} x_2^{64} = 4.153 \cdot 10^{92}$$

$$5.314 x_2^{64} = 4.153 \cdot 10^6$$

$$x_2^{64} = \frac{4.153 \cdot 10^6}{5.314}$$

$$5.314$$

$$\begin{aligned}
 \log 4.153 &= .618362 \\
 6 \log 10 &= 6. \\
 \text{Colog } 5.314 &= 9.274578-10 \\
 64 \log x_2 &= \underline{15.892940-10} \\
 \log x_2 &= .092077 \\
 x_2 &= 1.2362
 \end{aligned}$$

On substituting -1.2 we find it nearly satisfies the equation. On substituting 3.2 we find it nearly satisfies the equation. Then the real roots are

$$\begin{aligned}
 x_1 &= 3.2359 \\
 x_2 &= -1.2362
 \end{aligned}$$

The modulus for the pair of complex roots is found from the equation

$$-4.360 \cdot 10^{32} y^2 - .125 \cdot 10^{60} y - 5.314 \cdot 10^{86} = 0$$

when $y = x^{64}$

Dividing by the coefficient of y^2 we have

$$y^2 + \frac{.125 \cdot 10^{28}}{4.360} y + \frac{5.314 \cdot 10^{54}}{4.360} = 0$$

Now from the footnote on page 11 we see that the absolute term in the quadratic is equal to the square of the modulus of the complex roots. Let R be this modulus

then $(R = r^{64})$

$$R^2 = \frac{5.314 \cdot 10^{54}}{4.360}$$

$$\begin{aligned}
 \log 4.183 &= .61833 \\
 \log 10 &= 1.0 \\
 \log 0.834 &= .91979 \\
 \hline
 \log x_2 &= .53812 \\
 \log x_2 &= .53812 \\
 x_2 &= 3.44
 \end{aligned}$$

On substituting $x_2 = 3.44$ in the equation $x_1^2 + x_2^2 = 10$ we find $x_1 = 2.62$. Then the total tooth

$$\begin{aligned}
 x_1 &= 2.62 \\
 x_2 &= 3.44
 \end{aligned}$$

The modulus for the pair of complex roots is found

$$\begin{aligned}
 \text{From the equation} \\
 x^2 - 1.15x + 0.834 = 0 \\
 \text{when } x = 3.44
 \end{aligned}$$

dividing by the coefficient of x we have

$$x - 1.15 + \frac{0.834}{x} = 0$$

Now from the footnote on page 11 we see that the absolute term in the quadratic is equal to the square of the

$$\begin{aligned}
 R^2 &= 0.834 \\
 R &= 0.913
 \end{aligned}$$

Solving by logarithms

$$\log 5.314 = .725422$$

$$54 \log 10 = 54.000000$$

$$\text{colog } 4.360 = 9.360514 - 10$$

$$2 \log R = 54.085936$$

$$\log R (r^{64}) = 27.042968$$

$$\log r = .422547$$

$$r = 2.6457$$

Let the pair of complex roots be $u \pm Vi$, $u - Vi$.

Now since the sum of the roots of the equation is (-6), we have

$$x_1 + x_2 + 2u = -6$$

$$2u = -6 - 3.2359 + 1.2362i$$

$$u = -1.9999 + .6181i \text{ app.}$$

$$\text{Now } u^2 + v^2 = r^2$$

$$v^2 = r^2 - u^2 \quad \text{Substituting values of } u \text{ and } r$$

$$v = \sqrt{3}$$

The complex roots are therefore

$$-2 \pm \sqrt{3}i, -2 \mp \sqrt{3}i$$

We may check by finding the product of the roots and comparing with the constant term

$x_1 x_2 r^2 = 28.00153$ which is as near as we could expect. The middle terms of the quadratic fragments are the terms in which the sign is

Solving by iteration

$$\begin{aligned} \log 2.314 &= .36422 \\ 24 \log 10 &= 24.00000 \\ \text{error } 4.360 &= 2.360514-10 \\ \hline 2 \log 2 &= 24.360514 \\ \log 2 (10^{24}) &= 24.360514 \\ \log 2 &= .30103 \\ 2 &= 2.30103 \end{aligned}$$

Let the pair of complex roots be $u \pm vi$, $u - vi$.

Now since the sum of the roots of the equation is

$-(a)$, we have

$$\begin{aligned} x_1 + x_2 + 2u &= -a \\ 2u &= -4.6 - 2.360514 + 1.360514 \\ u &= -1.5 - 0.5 = -2.0 \end{aligned}$$

$$\text{Now } u^2 + v^2 = 4$$

Substituting values of u and v

$$v^2 = 4 - u^2$$

The complex roots are therefore

$$-2 \pm 1.732i, -2 - 1.732i$$

We may check by finding the product of the roots and

comparing with the constant term

$$x_1 x_2 + 2 = 28.00125 \text{ which is as near as we would}$$

expect.

TWO PAIR of COMPLEX ROOTS

If two pairs of complex roots are present the fluctuations in signs will occur in two columns. Let the roots of an equation be, $x_1, r_1 e^{i\theta_1}, r_1 e^{-i\theta_1}, r_2 e^{i\theta_2}, r_2 e^{-i\theta_2}$ and the order of magnitude to be

$|r_1| > |x_1| > |r_2|$. Then the equation is

$$(x - r_1 e^{i\theta_1})(x - r_1 e^{-i\theta_1})(x - x_1)(x - r_2 e^{i\theta_2})(x - r_2 e^{-i\theta_2}) = 0.$$

and the equation whose roots are the m^{th} powers of the roots of this equation will be

$$(y - r_1^m e^{im\theta_1})(y - r_1^m e^{-im\theta_1})(y - x_1^m)(y - r_2^m e^{im\theta_2})(y - r_2^m e^{-im\theta_2}) = 0, \text{ where } y = x^m$$

Expanding and taking the leading terms we have

$$y^5 - 2r_1^m \cos m\theta_1 y^4 + r_1^{2m} y^3 - x_1^m r_1^{2m} y^2 + 2x_1^m r_1^{2m} r_2^m \cos m\theta_2 y - x_1^m r_1^{2m} r_2^{2m} = 0.$$

The linear and quadratic fragments being

$$y^5 - 2r_1^m \cos m\theta_1 y^4 + r_1^{2m} y^3 = 0$$

$$r_1^{2m} y^3 - x_1^m r_1^{2m} y^2 = 0$$

$$-x_1^m r_1^{2m} y^2 + 2x_1^m r_1^{2m} r_2^m \cos m\theta_2 y - x_1^m r_1^{2m} r_2^{2m} = 0.$$

The roots of these fragments being the roots that we started with. The middle terms of the quadratic fragments are the terms in which the signs fluctuate.

$$x^6 + x^5 + 6x^4 - 20x^3 + 51x^2 - 53x - 130$$

	x^6	x^5	x^4	x^3	x^2	x^1	x^0
Given Equation	+1	+1	+6	-20	+51	-53	-130
	+1	-1.10 +1.2	+ .36 · 10 ² + .40 +1.02	-4.00 · 10 ² +6.12 +1.06 -2.60	+2.601 · 10 ³ -2.120 -1.560	- .281 · 10 ⁴ -1.326	+1.69 · 10 ⁴
2 nd Power	+1	+1.1 · 10	+1.78 · 10 ²	+ .58 · 10 ²	-1.079 · 10 ³	-1.607 · 10 ⁴	+1.69 · 10 ⁴
	+1	-1.21 · 10 ² +3.56	+3.168 · 10 ⁴ - .128 - .216	- .034 · 10 ⁵ -3.841 +3.535 + 338	+1.164 · 10 ⁶ +1.864 +6.016	-2.582 · 10 ⁸ - .365	+2.856 · 10 ⁸
4 th Power	+1	+2.35 · 10 ²	+2.824 · 10 ⁴	- .002 · 10 ⁵	+9.044 · 10 ⁶	-2.947 · 10 ⁸	+2.856 · 10 ⁸
	+1	-5.523 · 10 ⁴ +5.648	+7.975 · 10 ⁸ + .001 + .181	- .000 · 10 ¹¹ +5.106 +1.680 + .006	+8.179 · 10 ¹³ - .012 +1.613	-8.685 · 10 ¹⁶ + .517	+8.157 · 10 ¹⁶
8 th Power	+1	+ .125 · 10 ⁴	+8.157 · 10 ⁸	+6.792 · 10 ¹¹	+9.780 · 10 ¹³	-8.168 · 10 ¹⁶	+8.157 · 10 ¹⁶
	+1	- .016 · 10 ⁹ +1.631	+6.654 · 10 ¹⁷ - .017 + .002	-4.613 · 10 ²³ +1.596 + .002 + .000	+ .096 · 10 ²⁹ +1.109 + .001	-6.672 · 10 ³³ + .016	+6.654 · 10 ³³
16 th Power	+1	+1.615 · 10 ⁹	+6.639 · 10 ¹⁷	-3.015 · 10 ²³	+1.206 · 10 ²⁹	-6.656 · 10 ³³	+6.654 · 10 ³³
	+1	-2.608 · 10 ¹⁸ +1.328	+4.408 · 10 ³⁵ + .010 *	- .909 · 10 ⁴⁷ +1.601 *	+1.454 · 10 ⁵⁸ - .401 *	-4.430 · 10 ⁶⁷ *	+4.428 · 10 ⁶⁷
32 nd Power	+1	-1.280 · 10 ¹⁸	+4.418 · 10 ³⁵	+ .692 · 10 ⁴⁷	+1.053 · 10 ⁵⁸	-4.430 · 10 ⁶⁷	+4.428 · 10 ⁶⁷
	+1	-1.638 · 10 ³⁶ + .884	+1.952 · 10 ⁷¹ * *	- .479 · 10 ⁹⁴ + .930	+1.109 · 10 ¹¹⁶ + .061	-1.962 · 10 ¹³⁵	+1.961 · 10 ¹³⁵
64 th Power	+1	- .754 · 10 ³⁶	+1.952 · 10 ⁷¹	+ .431 · 10 ⁹⁴	+1.170 · 10 ¹¹⁶	-1.962 · 10 ¹³⁵	+1.961 · 10 ¹³⁵
	+1	-5.685 · 10 ⁷¹ +3.904	+3.810 · 10 ¹⁴² *	-1.858 · 10 ¹⁸⁷ +4.567	+1.369 · 10 ²³² + .002	-3.849 · 10 ²⁷⁰	+3.846 · 10 ²⁷⁰
128 th Power	+1	-1.781 · 10 ⁷¹	+3.810 · 10 ¹⁴²	+2.709 · 10 ¹⁸⁷	+1.371 · 10 ²³²	-3.849 · 10 ²⁷⁰	+3.846 · 10 ²⁷⁰

With this equation the fluctuations in signs will occur in the coefficients of y^4 and y . Other results hold as in case I for the same reasons.

In the illustrative example on page (21). We have broken the equation up into two linear and two quadratic fragments. The first real root is found from the linear fragment.

$$1.371 \cdot 10^{232} y = 3.849 \cdot 10^{270} \quad (y = x_1^{128})$$

$$x_1^{128} = \frac{3.849 \cdot 10^{38}}{1.371}$$

Then by logarithms $x_1 = 1.997$ (app. 2)

The second real root is found from the linear fragment

$$3.849 \cdot 10^{270} y = 3.846 \cdot 10^{270} \quad y = x_2^{128}$$

$$x_2^{128} = 1 \quad (\text{app.})$$

$$\log x_2 = 0$$

$$x_2 = 1$$

By substituting we find the signs to be $x_1 = 2$, $x_2 = -1$

The modulus of the first pair of complex roots is found

from the quadratic fragment $y^2 - 1.781 \cdot 10^{71} y + 3.810 \cdot 10^{142} = 0$.

as under example for one pair of complex roots

$$(r_1^{128})^2 = 3.810 \cdot 10^{142}$$

Solving by logarithms we find

$$r_1 = 3.6055$$

With this equation the fluctuations in signs will

occur in the coefficients of x_1 and x_2 . Other results hold as in case I for the same reasons.

In the illustrative example on page (21).

We have broken the equation up into two linear and two quadratic fragments. The first real root is found from the linear fragment.

$$1.371 \cdot 10^{-10} x_1^2 + 5.448 \cdot 10^{-10} x_1 + 1.371 \cdot 10^{-10} = 0$$

$$x_1 = \frac{-5.448 \cdot 10^{-10} \pm \sqrt{(5.448 \cdot 10^{-10})^2 - 4 \cdot 1.371 \cdot 10^{-10} \cdot 1.371 \cdot 10^{-10}}}{2 \cdot 1.371 \cdot 10^{-10}}$$

Then by formula $x_1 = 1.371 \cdot 10^{-10}$ (approx.)

The second real root is found from the linear fragment

$$5.849 \cdot 10^{-10} x_2 + 5.848 \cdot 10^{-10} = 0$$

$$x_2 = -1 \text{ (approx.)}$$

$$10^4 x_3 = 0$$

$$x_4 = 1$$

By substituting we find the signs to be $x_1 = +$, $x_2 = -$, $x_3 = -$, $x_4 = +$

The modulus of the first pair of complex roots is found

$$\text{from the quadratic fragment } x_1^2 - 1.731 \cdot 10^{-10} x_1 + 3.810 \cdot 10^{-10} = 0$$

as under example for one pair of complex roots

$$r_1 = 3.810 \cdot 10^{-10}$$

Solving by formula we find

$$r_1 = 3.810 \cdot 10^{-10}$$

The modulus of the second pair of complex roots is found from the quadratic fragment

$$3.810 \cdot 10^{142} y^2 + 2709 \cdot 10^{187} y + 1.371 \cdot 10^{232} = 0$$

$$y^2 + \frac{2.709 \cdot 10^{187}}{3.810 \cdot 10^{142}} y + \frac{1.371 \cdot 10^{232}}{3.810 \cdot 10^{142}} = 0$$

then $(r_2^{128})^2 = \frac{1.371 \cdot 10^{132}}{3.810 \cdot 10^{142}}$

by logarithms

$$r_2 = 2.238$$

We will now let the complex roots be

$$a_1 + b_1 i, a_1 - b_1 i, a_2 + b_2 i, a_2 - b_2 i,$$

Then the sum of the roots is

$$x_1 + x_2 + 2a_1 + 2a_2 = -1 \quad (x_1 + x_2 = -1)$$

(I) $a_1 + a_2 = -1$

Now applying the theorem on page 15 for the reciprocals of the roots

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{a_1 + b_1 i} + \frac{1}{a_1 - b_1 i} + \frac{1}{a_2 + b_2 i} + \frac{1}{a_2 - b_2 i} = -\frac{53}{130}$$

rationalizing the denominators and substituting

$$a_1^2 + b_1^2 = r_1^2 \text{ and } a_2^2 + b_2^2 = r_2^2$$

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{2a_1}{r_1^2} + \frac{2a_2}{r_2^2} = -\frac{53}{130}$$

The modulus of the second pair of complex roots is found

from the quadratic equation

$$\begin{aligned}
 & 2.810 \cdot 10^{-10} x^2 + 2.703 \cdot 10^{-10} x + 1.371 \cdot 10^{-10} = 0 \\
 & x_1 = \frac{-2.703 \cdot 10^{-10} \pm \sqrt{(2.703 \cdot 10^{-10})^2 - 4 \cdot 2.810 \cdot 10^{-10} \cdot 1.371 \cdot 10^{-10}}}{2 \cdot 2.810 \cdot 10^{-10}} \\
 & \text{then } |x_2| = \frac{1.371 \cdot 10^{-10}}{2.810 \cdot 10^{-10}}
 \end{aligned}$$

by logarithms

$$r_2 = 2.338$$

We will now let the complex roots be

$$a_1 + b_1 i, a_1 - b_1 i, a_2 + b_2 i, a_2 - b_2 i$$

Then the sum of the roots is

$$x_1 + x_2 + x_3 + x_4 = -1 \quad (x_1 + x_2 = -1)$$

$$(1) \quad a_1 + a_2 = -1$$

Now applying the theorem on page 12 for the reciprocals

of the roots

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = \frac{1}{a_1 + b_1 i} + \frac{1}{a_1 - b_1 i} + \frac{1}{a_2 + b_2 i} + \frac{1}{a_2 - b_2 i} = -\frac{22}{120}$$

rationalizing the denominators and substituting

$$a_1 + b_1 i = r_1 e^{i\theta_1} \text{ and } a_2 + b_2 i = r_2 e^{i\theta_2}$$

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = -\frac{22}{120}$$

now the values substituted and dividing by 2

$$\frac{1}{x_1} = .5, \quad \frac{1}{x_2} = -1, \quad \frac{1}{r_1^2} = .0769, \quad \frac{1}{r_2^2} = .1997 \text{ (.2 app)}$$

gives

$$.0769 a_1 \neq .2 a_2 = .0462$$

$$\text{now } a_1 \neq a_2 = -1 \text{ (I)}$$

Solving simultaneously

$$a_1 = -2, \quad a_2 = 1$$

$$\text{now } a_1^2 \neq b_1^2 = r_1^2 \text{ and } a_2^2 \neq b_2^2 = r_2^2$$

substituting and solving for b_1 and b_2

$$b_1 = 3, \quad b_2 = 2$$

Then the two pair of complex roots are

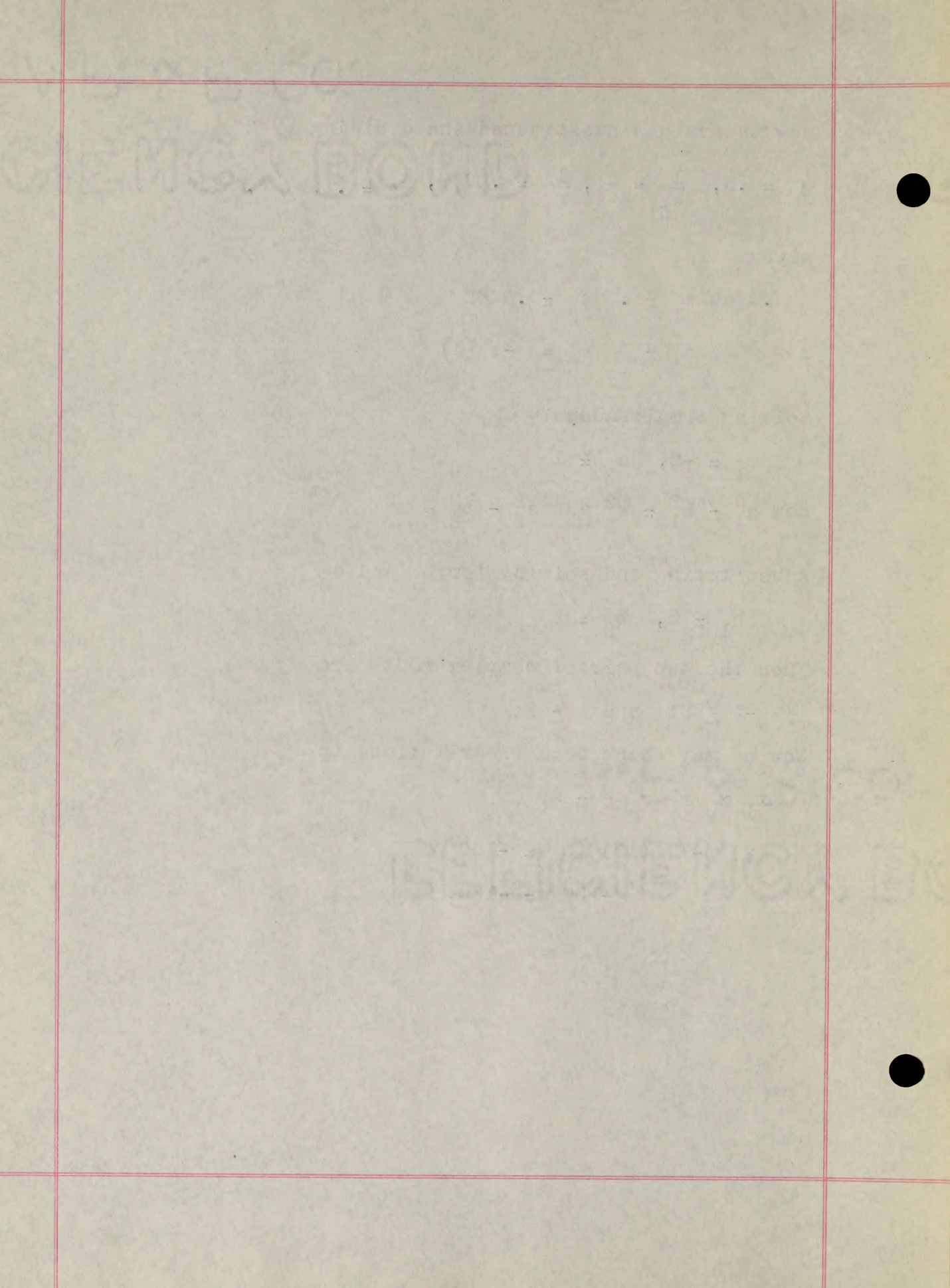
$$-2 \pm 3i \quad \text{and} \quad 1 \pm 2i$$

Now we may check from the relationship

$$x_1 x_2 r_1^2 r_2^2 = -130$$

$$(2) (-1) (13) (5) = -130$$

$$-130 = -130.$$



ONE PAIR of COINCIDENT COMPLEX ROOTS (Original)

Let a pair of coincident complex roots be $(r e^{i\theta})^2$, $(r e^{-i\theta})^2$ and the order of magnitude of the roots be $|r| > |x_1| > |x_2|$ of the equation $f(x) = 0$. That is

$$(x - r e^{i\theta})^2 (x - r e^{-i\theta})^2 (x - x_1) (x - x_2) = 0.$$

and the equation whose roots are the m^{th} powers of these roots is

$$(y - r^m e^{im\theta})^2 (y - r^m e^{-im\theta})^2 (y - x_1^m) (y - x_2^m) = 0.$$

On multiplying and taking the leading coefficients we have

$$y^6 - 4r^m \cos m\theta y^5 + 4r^{2m} y^4 - 4r^{3m} \cos m\theta y^3 + r^{4m} y^2 - x_1^m r^{4m} y + x_1^m x_2^m r^{4m} = 0.$$

The fragments are therefore

$$\text{I } y^6 - 4r^m \cos m\theta y^5 + 4r^{2m} y^4 = 0$$

$$\text{II } 4r^{2m} y^4 - 4r^{3m} \cos m\theta y^3 + r^{4m} y^2 = 0$$

$$r^{4m} y^2 - x_1^m r^{4m} y = 0$$

$$-x_1^m r^{4m} + x_1^m x_2^m r^{4m} = 0$$

Now regard I as $(y - 2r^m)^2$ and II as $(2y - r^{2m})^2$

Finding the equation whose roots are the $2m^{\text{th}}$ powers of the given roots we get

$$y^6 - 8r^{2m} \cos 2m\theta y^5 + (18r^{4m} - 32 r^{4m} \cos^3 m\theta) y^4 - 8r^{6m} \cos 2m\theta y^3 + r^{8m} y^2 - x_1^{2m} r^{8m} y + x_1^{2m} x_2^{2m} r^{8m} = 0.$$

ONE PAIR OF COMPLEX CONJUGATE ROOTS (continued)

Let a pair of coincident complex roots be $(r \pm i\alpha)^2$.
 If $r \neq 0$, and the order of multiplicity of the roots be
 $|r| > |x| > |x|$ of the equation $f(x) = 0$. That is

$$(x - r - i\alpha)^2 (x - r + i\alpha)^2 (x - r_1) \dots (x - r_n) = 0.$$

and the equation whose roots are the m th powers of these
 roots is

$$(y - r^m - i\alpha^m)^2 (y - r^m + i\alpha^m)^2 (y - r_1^m) \dots (y - r_n^m) = 0.$$

On multiplying and taking the leading coefficients we have

$$y^4 - 2r^m y^3 + (r^4 + \alpha^4) y^2 - 2r^m \alpha^4 y + r^4 - \alpha^4 = 0.$$

$$-2r^m \alpha^4 y + r^4 - \alpha^4 = 0.$$

The discriminant of this equation

$$y^4 - 2r^m y^3 + (r^4 + \alpha^4) y^2 - 2r^m \alpha^4 y + r^4 - \alpha^4 = 0$$

$$y^4 - 2r^m y^3 + (r^4 + \alpha^4) y^2 - 2r^m \alpha^4 y + r^4 - \alpha^4 = 0$$

$$y^4 - 2r^m y^3 + (r^4 + \alpha^4) y^2 - 2r^m \alpha^4 y + r^4 - \alpha^4 = 0$$

$$y^4 - 2r^m y^3 + (r^4 + \alpha^4) y^2 - 2r^m \alpha^4 y + r^4 - \alpha^4 = 0$$

Whence the equation whose roots are the m th powers of

the given roots is

$$y^4 - 2r^m y^3 + (r^4 + \alpha^4) y^2 - 2r^m \alpha^4 y + r^4 - \alpha^4 = 0$$

$$y^4 - 2r^m y^3 + (r^4 + \alpha^4) y^2 - 2r^m \alpha^4 y + r^4 - \alpha^4 = 0.$$

Now taking fragment I

$$y^6 - 4r^m \cos(m \cdot \theta) y^5 + 4r^{2m} y^4 = 0$$

$$y^4 (y^2 - 4r^m \cos(m \cdot \theta) y + 4r^{2m}) = 0$$

$$y^2 - (2r^m) \left[\{ \cos m \cdot \theta + i \sin m \cdot \theta \} + \{ \cos m \cdot \theta - i \sin m \cdot \theta \} \right] y + (2r^m)^2 = 0$$

$$y^2 - (2r^m) (e^{im\theta} + e^{-im\theta}) y + (2r^m)^2 = 0$$

$$(y - 2r^m e^{im\theta}) (y - 2r^m e^{-im\theta}) = 0$$

$$y = 2r^m e^{im\theta}, \quad y = 2r^m e^{-im\theta}$$

Then the constant term of the first quadratic fragment is $4r^{2m}$.

For the second quadratic fragment

$$4r^{2m} y^4 - 4r^{3m} \cos(m \cdot \theta) y^3 + r^{4m} y^2 = 0$$

$$r^{2m} y^2 (4y^2 - 4r^m \cos m \cdot \theta y + r^{2m}) = 0$$

$$y^2 - r^m \cos m \cdot \theta y + \frac{r^{2m}}{4} = 0.$$

$$y^2 - \frac{r^m}{2} \left[\{ \cos m \cdot \theta + i \sin m \cdot \theta \} + \{ \cos m \cdot \theta - i \sin m \cdot \theta \} \right] y + \left(\frac{r^m}{2} \right)^2 = 0.$$

$$y^2 - \frac{r^m}{2} (e^{im\theta} + e^{-im\theta}) y + \left(\frac{r^m}{2} \right)^2 = 0$$

$$\left(y - \frac{r^m}{2} e^{im\theta} \right) \left(y - \frac{r^m}{2} e^{-im\theta} \right) = 0. \quad y = \frac{r^m e^{\pm im\theta}}{2}$$

Then the constant term of the second quadratic fragment is $\frac{r^{2m}}{4}$.

Now $4r^{2m}$ is the last term of fragment (I) and the first term of fragment (II). The roots of fragment (I) are twice the m^{th} powers of the original complex roots and the roots of fragment (II) are one half the m^{th} powers of the original complex roots. Therefore the roots of these fragments are

Now taking fragment 1

$$y^2 - 4r^2 \cos(m \cdot \theta) y + 4r^{2m} = 0$$

$$y^2 - 4r^2 \cos(m \cdot \theta) y + 4r^{2m} = 0$$

$$y^2 - (2r^m) \left[\cos m \cdot \theta + i \sin m \cdot \theta \right] + \left\{ \cos m \cdot \theta - i \sin m \cdot \theta \right\} y + 4r^{2m} = 0$$

$$y^2 - (2r^m) (e^{im\theta} + e^{-im\theta}) y + 4r^{2m} = 0$$

$$(y - 2r^m e^{im\theta}) (y - 2r^m e^{-im\theta}) = 0$$

$$y = 2r^m e^{im\theta}, \quad y = 2r^m e^{-im\theta}$$

Then the constant term of the first quadratic fragment

$$is 4r^{2m}$$

for the second quadratic fragment

$$4r^{2m} y^2 - 4r^{2m} \cos(m \cdot \theta) y + r^{4m} = 0$$

$$4r^{2m} y^2 - 4r^{2m} \cos(m \cdot \theta) y + r^{4m} = 0$$

$$y^2 - r^2 \cos m \cdot \theta y + \frac{r^{4m}}{4} = 0$$

$$y^2 - \frac{r^{2m}}{2} \left[\cos m \cdot \theta + i \sin m \cdot \theta \right] + \left\{ \cos m \cdot \theta - i \sin m \cdot \theta \right\} y + \frac{r^{4m}}{4} = 0$$

$$y^2 - \frac{r^{2m}}{2} (e^{im\theta} + e^{-im\theta}) y + \frac{r^{4m}}{4} = 0$$

$$(y - \frac{r^{2m}}{2} e^{im\theta}) (y - \frac{r^{2m}}{2} e^{-im\theta}) = 0, \quad y = \frac{r^{2m} e^{im\theta}}{2}, \quad y = \frac{r^{2m} e^{-im\theta}}{2}$$

Then the constant term of the second quadratic fragment

$$is \frac{r^{4m}}{4}$$

Now $4r^{2m}$ is the last term of fragment (1) and the first

term of fragment (2). The roots of fragment (1) are twice

the m^{th} powers of the original complex roots and the roots of

fragment (2) are one half the m^{th} powers of the original

complex roots. Therefore the roots of these fragments are

functionally related by the intermediate term ($4r^{2m}$) and therefore will not disappear as the root squaring process is continued.

In the transformed equation whose roots are the $2m^{\text{th}}$ powers of the original roots we see that this intermediate term does not disappear.

In the illustrative problem the coefficient between the coefficient fluctuating in sign indicates coincident roots. Solving under this supposition for the modulus "r" we find that the moduli are identical.

Of course it would be possible to solve for the multiple roots by finding the H. C. F. of $f(x) = 0$ and $f'(x) = 0$. I do not believe though that this test would be tried first if an equation was to be solved. If the root square method was tried first and there were coincident roots it would be a great deal easier to continue and find all roots by this method, provided all roots are desired.

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Of course it would be possible to solve for the multiple roots by finding the H. C. F. of $f(x) = 0$ and $f'(x) = 0$. I do not believe though that this would be tried first if an equation was to be solved. If the root square method was tried first and there were coincident roots it would be a great deal easier to continue and find all roots by this method, provided all roots are desired.

	y^6	y^5	y^4	y^3
m^{th} Power	1	$-4r^m \cos m\theta$	$\sqrt{4}r^{2m}$	$-4r^{3m} \cos m\theta$
	$\sqrt{1}$	$-16r^{2m} \cos^2 m\theta$ $8r^{2m} n$	$\sqrt{16}r^{4m}$ $-32r^{4m} \cos^2 m\theta c$ $2r^{4m} n$	$-16r^{6m} \cos^2 m\theta$ $8r^{6m} n$ $-8r^{5m} x_1^m \cos m\theta c$ $2x_1^{2m} x_2^{2m} r^{8m} n$
$2m^{th}$ Power	1	$-16r^{2m} \cos^2 m\theta$ $\sqrt{8}r^{2m}$	$18r^{4m}$ $-32r^{4m} \cos^2 m\theta$	$-16r^{6m} \cos^2 m\theta$ $8r^{6m}$
y^2		y^1	y^0	
\sqrt{r}^{4m}		$-x_1^m r^{4m}$	$\sqrt{x_1^m x_2^m} r^{4m}$	
\sqrt{r}^{8m} $-8x_1^m r^{7m} \cos m\theta$ $\sqrt{8}x_1^m x_2^m r^{6m}$		$-x_1^{2m} r^{8m}$ $\sqrt{2}x_1^m x_2^m r^{8m} n$	$\sqrt{x_1^{2m} x_2^{2m}} r^{8m}$	
\sqrt{r}^{8m}		$-x_1^{2m} r^{8m}$	$\sqrt{x_1^{2m} x_2^{2m}} r^{8m}$	

$$4x^6 + 4x^4 + 36x^3 - 39x^2 + 70x - 25 = 0$$

$$x^6 + x^4 + 9x^3 - 9.75x^2 + 17.5x - 6.25 = 0$$

	x^6	x^5	x^4	x^3	x^2	x^1	x^0
Given Equation	+1	+0	+1	+9	-9.75	+17.5	-6.25
	+1	-0 +2 η	+ .10 · 10 +0 -1.95 η	-8.10 · 10 -1.95 +0 -1.25 η	+ .951 · 10 ² -3.150 - .125 η	-3.063 · 10 ² +1.219 η	+3.906 · 10
2 nd Power	+1	+2	-1.85 · 10	-1.13 · 10 ²	-2.324 · 10 ²	-1.844 · 10 ²	+3.906 · 10
	+1	- .4 · 10 - 3.7 η	+3.423 · 10 ² +4.52 -4.648 η	-1.277 · 10 ⁴ + .860 + .074 + .008 η	+5.401 · 10 ⁴ -4.167 - .145 η	-3.400 · 10 ⁴ -1.816 η	+1.526 · 10 ³
4 th Power	+1	-4.10 · 10	+3.295 · 10 ²	- .335 · 10 ⁴	+1.089 · 10 ⁴	-5.216 · 10 ⁴	+1.526 · 10 ³
	+1	-1.681 · 10 ³ + .659 η	+1.086 · 10 ⁵ -2.747 + .218 η	-1.222 · 10 ⁷ + .717 - .428 + .000 η	+1.186 · 10 ⁸ -3.495 + .010 η	-2.721 · 10 ⁹ + .033 η	+2.329 · 10 ⁶
8 th Power	+1	-1.022 · 10 ³	-1.443 · 10 ⁵	- .833 · 10 ⁷	-2.299 · 10 ⁸	-2.688 · 10 ⁹	+2.329 · 10 ⁶
	+1	-1.044 · 10 ⁶ - .289 η	+2.082 · 10 ¹⁰ -1.703 - .046 η	-6.939 · 10 ¹³ +6.634 - .549 η	+5.285 · 10 ¹⁶ -4.478 - * η	-7.225 · 10 ¹⁸ - .001 η	+5.424 · 10 ¹²
16 th Power	+1	-1.333 · 10 ⁶	+3.330 · 10 ⁹	-8.540 · 10 ¹²	+8.070 · 10 ¹⁵	-7.224 · 10 ¹⁸	+5.424 · 10 ¹²
	+1	-1.777 · 10 ¹² + .007 η	+1.089 · 10 ¹⁹ -2.277 + .002 η	-7.293 · 10 ²⁵ +5.374 -1.926 η	+6.512 · 10 ³¹ -12.339 * η	-5.219 · 10 ³⁷ * η	+2.942 · 10 ²⁵
32 nd Power	+1	-1.770 · 10 ¹²	-1.186 · 10 ¹⁹	-3.845 · 10 ²⁵	-5.827 · 10 ³¹	-5.219 · 10 ³⁷	+2.942 · 10 ²⁵
	+1	-3.133 · 10 ²⁴ * η	+1.407 · 10 ³⁸ -1.361 η	-1.478 · 10 ⁵¹ +1.382 - .185 η	+3.395 · 10 ⁶³ -4.013 η	-2.724 · 10 ⁷⁵	+8.655 · 10 ⁵⁰
64 th Power	+1	-3.133 · 10 ²⁴	+ .046 · 10 ³⁸	- .281 · 10 ⁵¹	- .618 · 10 ⁶³	-2.724 · 10 ⁷⁵	+8.655 · 10 ⁵⁰
	+1	-9.816 · 10 ⁴⁸	+ .021 · 10 ⁷⁵ -1.761 η	-7.896 · 10 ¹⁰⁰ - .448 +1.707 η	+ .382 · 10 ¹²⁶ -1.531 η	-7.420 · 10 ¹⁵⁰	+7.491 · 10 ¹⁰¹
128 th Power	+1	-9.816 · 10 ⁴⁸	-1.740 · 10 ⁷⁵	-6.637 · 10 ¹⁰⁰	-1.149 · 10 ¹²⁶	-7.420 · 10 ¹⁵⁰	+7.491 · 10 ¹⁰¹
	+1	-9.635 · 10 ⁹⁷	+3.028 · 10 ¹⁵⁰ +1.303 η	-4.405 · 10 ²⁰¹ -3.998 - .146 η	+1.320 · 10 ²⁵² - .985 η	-5.506 · 10 ³⁰¹	+5.612 · 10 ²⁰³
256 th Power	+1	-9.635 · 10 ⁹⁷	+4.331 · 10 ¹⁵⁰	-8.549 · 10 ²⁰¹	+ .335 · 10 ²⁵²	-5.506 · 10 ³⁰¹	+5.612 · 10 ²⁰³

Taking the equation whose roots are the 32^{nd} powers of the roots of the original equations, since if we have coincident roots the roots are widely separated at this point.

$$1.770 \cdot 10^{12} y^2 + 1.186 \cdot 10^{19} y + 3.845 \cdot 10^{25} = 0$$

$$4R_1^2 = \frac{3.845 \cdot 10^{13}}{1.770} \quad (1^{\text{st}} \text{ quadratic fragment})$$

$$R_1^2 = \frac{3.845 \cdot 10^{13}}{7.080} \quad R_1 = r_1^{64}$$

$$\log 3.845 = .58490$$

$$13 \log 10 = 13.$$

$$\text{colog } 7.080 = \underline{9.14997-10}$$

$$64 \log r_1 = 12.73487$$

$$\log r_1 = .19898$$

$$r_1 = 1.5812$$

$$+3.845 \cdot 10^{25} y^2 + 5.827 \cdot 10^{31} y + 5.219 \cdot 10^{37} = 0.$$

$$\frac{R_2}{4} = \frac{5.219 \cdot 10^{12}}{3.845} \quad (2^{\text{nd}} \text{ quadratic fragment})$$

$$R_2 = \frac{20.876 \cdot 10^{12}}{3.845}$$

$$\log 20.876 = 1.31964$$

$$12 \log 10 = 12.$$

$$\text{colog } 3.845 = \underline{9.41510-10}$$

$$64 \log r_2 = 12.73474$$

$$\log r_2 = .19898$$

$$r_2 = 1.5812$$

Now since $r_1 = r_2$ we have a pair of coincident roots.

Taking the equation whose roots are the 2nd powers of the

roots of the original equation, since if we have coinc-

ident roots the roots are widely separated at this point.

$$1.770 \cdot 10^{12} x^2 + 1.155 \cdot 10^{12} x + 3.845 \cdot 10^{22} = 0$$

$$R_1 = \frac{3.845 \cdot 10^{12}}{1.770} \quad (1^{st} \text{ quadratic fragment})$$

$$R_1 = \frac{3.845 \cdot 10^{12}}{1.770} \quad R_1 = r_1$$

$$\log 3.845 = 0.5850$$

$$12 \log 10 = 12$$

$$\text{color } 7.930 = 0.1457 - 10$$

$$\log 10^7 = 0.7347$$

$$\log r_1 = 1.3203$$

$$r_1 = 1.5812$$

$$13.845 \cdot 10^{22} x^2 + 1.327 \cdot 10^{21} x + 3.219 \cdot 10^{27} = 0$$

$$R_2 = \frac{3.219 \cdot 10^{12}}{1.327} \quad (2^{nd} \text{ quadratic fragment})$$

$$R_2 = \frac{3.219 \cdot 10^{12}}{1.327}$$

$$\log 3.219 = 1.5074$$

$$12 \log 10 = 12$$

$$\text{color } 3.845 = 0.4151 - 10$$

$$\log 10^7 = 0.7347$$

$$\log r_2 = 1.9923$$

$$r_2 = 1.5812$$

Now since $r_1 = r_2$ we have a pair of coincident roots.

Solving for the real roots:

$$y^1 = 1.770 \cdot 10^{12}$$

$$x_1^{32} = 1.770 \cdot 10^{12}$$

$$\log 1.770 = .24797$$

$$12 \log 10 = 12.00000$$

$$32 \log x_1 = 12.24797$$

$$\log x_1 = .38275$$

$$x_1 = 2.414$$

$$5.219 \cdot 10^{37} y_2 = 2.942 \cdot 10^{25}$$

$$\begin{aligned} x_2^{32} &= \frac{2.942}{5.219 \cdot 10^{12}} \end{aligned}$$

$$\log 2.942 = .46864$$

$$\text{colog } 5.219 = 9.28241 - 10$$

$$\text{colog } 10^{12} = 8.00000 - 20$$

$$32 \log x_2 = 7.75105 - 20$$

$$\log x_2 = 9.61722 - 10$$

$$x_2 = .4142$$

checking the real roots for the signs we find

$$x_1 = -2.414, \quad x_2 = .4142$$

Now taking the equation for the reciprocals of the roots and letting the complex roots be $a + bi$, $a - bi$

$$\frac{1}{x_1} + \frac{1}{x_2} + 2 \left(\frac{1}{a + bi} \right) + 2 \left(\frac{1}{a - bi} \right) = \frac{70}{25}$$

Solving for the real roots:

$$x_1 = 1.770 \cdot 10^{12}$$

$$x_2 = 1.770 \cdot 10^{12}$$

$$\log 1.770 = .24787$$

$$12 \log 10 = 12.00000$$

$$12 \log x_1 = 12.24787$$

$$\log x_1 = .33875$$

$$x_1 = 2.414$$

$$2.219 \cdot 10^{12} = 2.242 \cdot 10^{12}$$

$$= 2.242$$

$$2.219 \cdot 10^{12}$$

$$\log 2.219 = .34534$$

$$\log 2.219 = .34534$$

$$\log 10 = 1.00000$$

$$12 \log x_2 = 12.34534$$

$$\log x_2 = .34534$$

$$x_2 = .4142$$

checking the real roots for the signs we find

$$x_1 = -2.414, x_2 = .4142$$

Now taking the equation for the reciprocals of the roots

and letting the complex roots be $a \pm bi$, $a - bi$

$$\frac{1}{x_1} + \frac{1}{x_2} + 2 \left(\frac{1}{a^2 + b^2} \right) + 2 \left(\frac{1}{a - bi} \right) = \frac{70}{25}$$

Rationalizing the denominators and placing $a^2 + b^2 = r^2$

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{4a}{r^2} = \frac{70}{25}$$

Now substituting the numerical values

$$\frac{1}{x_1} = .4143 \quad \frac{1}{x_2} = 2.4143 \quad \frac{1}{r^2} = .4$$

1.6 a = .8 respect to x. These are evaluated for

a = .5 by MacLaurin's formula the expansion of y

Then the coincident roots are

$$.5 \pm 1.5812i, .5 - 1.5812i$$

We may check by taking the negative sum of the roots which should be zero. We see that the sum is zero and the roots check.

a = 1 and substituting these values in the series will give the roots of f(y) = 0, providing these series are convergent.

If in placing the x in all terms but the first and last and expanding in MacLaurin's formula we find some of the series divergent, it becomes necessary to substitute the x in other terms including the first or last to get converging series. By varying these substitutions the several convergent series giving the roots will be found.

In the illustrative examples we have a quartic which gives converging series for all of the roots when x is placed in all of the terms but the first and last.

Rationalizing the denominators and placing a $\frac{1}{x^2} = x^{-2}$

$$\frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{x^2} = \frac{70}{x^2}$$

Now substituting the numerical values

$$\frac{1}{x^2} = .4143 \quad \frac{1}{x^2} = 2.4143 \quad \frac{1}{x^2} = .4$$

$$1.6 = .6$$

$$2 = .2$$

Then the coincident roots are

$$2 - 1.5812, \quad 2 - 1.5812$$

We may check by taking the negative sum of the roots

which should be zero. We see that the sum is zero and

the roots check.

SOLUTION BY MACLAURIN'S FORMULA

Given $f(y) = 0$ we may solve by expanding in MacLaurin's formula as follows:

If we place an x in all the terms of $f(y) = 0$ with the exception of the first and last, y becomes an implicit function of x . We may then obtain the successive derivatives of y in respect to x . These are evaluated for $x = 0$. Then by MacLaurin's formula the expansion of y in terms of x is known.

$$y = y_0 + \left(\frac{dy}{dx}\right)_0 x + \left(\frac{d^2y}{dx^2}\right)_0 \frac{x^2}{2!} + \left(\frac{d^3y}{dx^3}\right)_0 \frac{x^3}{3!} + \dots$$

y_0 are the values found for "y" when "x" is placed equal to zero and the resulting binomial equations are solved. The values of y_0 substituted in $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc.,

gives the values of $\left(\frac{dy}{dx}\right)_0$, $\left(\frac{d^2y}{dx^2}\right)_0$ etc., now letting

$x = 1$ and substituting these values in the series will give the roots of $f(y) = 0$, providing these series are convergent.

If in placing the x in all terms but the first and last and expanding in MacLaurin's formula we find some of the series divergent, it becomes necessary to substitute the x in other terms including the first or last to get converging series. By varying these substitutions the several convergent series giving the roots will be found.

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SOLUTION BY MACLAURIN'S FORMULA

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$$y = y_0 + \left(\frac{dy}{dx}\right)_0 x + \frac{1}{2!} \left(\frac{d^2y}{dx^2}\right)_0 x^2 + \frac{1}{3!} \left(\frac{d^3y}{dx^3}\right)_0 x^3 + \dots$$

in terms of x is known. y_0 are the values found for "y" when "x" is placed equal to zero and the resulting binomial equations are solved. The values of y_0 substituted into $\frac{dy}{dx}$, etc., gives the values of $\left(\frac{dy}{dx}\right)_0$, $\left(\frac{d^2y}{dx^2}\right)_0$, etc., now letting $x = 1$ and substituting these values in the series will give the roots of $f(y) = 0$, providing these series are convergent.

If in placing the x in all terms but the first and last and expanding in MacLaurin's formula we find some of the series divergent, it becomes necessary to substitute the x in other terms and when the first or last to get converging series. By varying these substitutions the several convergent series giving the roots will be found.

In the illustrative examples we have a quartic which gives converging series for all of the roots when x is placed in all of the terms but the first and last.

and a quartic where the series are divergent when x is placed in these terms. In this case by first placing x in the constant term we have two converging series giving two roots and again placing x in the leading term we have two other converging series giving the other two roots.

Solve:

$$y^4 - 7y^2 \neq 474y - 14.040 = 0$$

Introducing an x in all terms but the first and last

$$y^4 - 7y^2x \neq 474yx - 14,040 = 0.$$

$$\frac{dy}{dx} = \frac{7y^2 - 474y}{4y^3 - 14yx \neq 474x}$$

$$\text{Let } N = 7y^2 - 474y$$

$$D = 4y^3 - 14yx - 474x$$

$$\frac{d^2y}{dx^2} = \frac{D(14y - 474) \frac{dy}{dx} - N [(12y^2 - 14x) \frac{dy}{dx} - 14y \neq 474]}{D^2}$$

now if $x = 0$

$$y = 10.89 \quad -10.89 \quad 10.89i \quad -10.89i$$

$$\frac{dy}{dx} = -.84 \quad -1.16 \quad 1-.16i \quad 1/.16i$$

$$\frac{d^2y}{dx^2} = -.04 \quad \neq .08 \quad 0.00/.06i \quad 0.00-.06i$$

Now substituting these values in MacLaurin's formula where

$$x = 1$$

$$y = y_0 \neq \left(\frac{dy}{dx}\right)_0 x \neq \left(\frac{d^2y}{dx^2}\right)_0 \frac{x^2}{2} + \dots$$

and evaluating the four series we have

$$y_1 = 10.03 \quad y_3 = 1 + 10.76i$$

$$y_2 = -11.97 \quad y_4 = 1 - 10.76i$$

$$y_1 y_2 y_3 y_4 = 14.134 \quad (\text{check})$$

Solve:

$$y^4 + 13y^2 + 12 = 0 \quad * (\text{Page 35})$$

$$y^4 + 13y^2 + 12x = 0$$

Placing x in the constant term

$$\frac{dy}{dx} = \frac{-6}{2y^3 + 13y}$$

$$\frac{d^2y}{dx^2} = \frac{6(6y^2 + 13) \frac{dy}{dx}}{(2y^3 + 13y)^2}$$

when x = 0

$$y = 3.605i \quad -3.605i$$

$$\frac{dy}{dx} = -.127i \quad +.127i$$

$$\frac{d^2y}{dx^2} = -.032i \quad +.032i$$

Substituting in MacLaurin's series and evaluating

$$y_1 = 3.462i, \quad y_2 = -3.462i$$

Now placing x in the first term

$$xy^4 + 13y^2 + 12 = 0.$$

$$\frac{dy}{dx} = \frac{-y^4}{26y - 4xy^3}$$

$$\frac{d^2y}{dx^2} = \frac{-4y^3 (26y - 4xy^3) \frac{dy}{dx} + y^4 [(26 - 12xy^2) \frac{dy}{dx} - 4y^3]}{(26y - 4xy^3)^2}$$

when x = 0.

and evaluating the four series we have

$$y_1 = 10.03 \quad y_2 = 1 - 10.03$$

$$y_3 = -11.97 \quad y_4 = 1 - 10.03$$

$$y_1 y_3 y_4 = 14.134 \quad (\text{check})$$

Solve;

$$y^2 + 13y + 12 = 0 \quad (\text{Page 35})$$

$$y^2 + 13y + 12x = 0$$

Place x in the constant term

$$\frac{dy}{dx} = \frac{-y}{2y^2 + 13y}$$

$$\frac{dy}{y} = \frac{-1}{2y + 13} \frac{dy}{dx}$$

when $x = 0$

$$y = 2.0031 \quad y = -8.0031$$

$$\frac{dy}{dx} = -1.001 \quad \frac{dy}{dx} = 1.001$$

$$\frac{dy}{dx} = -0.001 \quad \frac{dy}{dx} = 0.001$$

Substitute in the first series and evaluate

$$y_1 = 0.0031, \quad y_2 = -0.0031$$

Now place x in the first term

$$xy^2 + 13y + 12 = 0$$

$$\frac{dy}{dx} = \frac{-y}{2y^2 + 13y}$$

$$\frac{dy}{y} = \frac{-1}{2y + 13} \frac{dy}{dx}$$

when $x = 0$

$$\begin{aligned} y &= .9607 i & - .9607 i \\ \frac{dy}{dx} &= .0341 i & - .0341 i \\ \frac{d^2y}{dx^2} &= .0057 i & - .0057 i \end{aligned}$$

Substituting and evaluating

$$\begin{aligned} y_3 &= .9977 i, & y_4 &= -.9977 i \\ y_1 y_2 y_3 y_4 &= 11.93 \text{ (check)} \end{aligned}$$

*

If we solve in the usual way

$$y^4 + 13x y^2 + 12 = 0$$

$$\frac{dy}{dx} = \frac{-13y^2}{4y^3 + 26x y}$$

$$\frac{d^2y}{dx^2} = \frac{(52y^4 - 676x y + 338y^2 x) \frac{dy}{dx} + 338y^3}{(4y^3 + 26x y)^2}$$

when $x = 0$

$$y = \sqrt[4]{3} + i\sqrt[4]{3} = 1.32 + 1.32i$$

$$\frac{dy}{dx} = 2.28 - 2.28i$$

$$\frac{d^2y}{dx^2} = 162.9 + 0(1)$$

We see that the real part of the series gives a divergent series and therefore the entire series is divergent.

$$\begin{aligned} y &= -0.00071 \\ \frac{dy}{dx} &= 0.00411 \\ \frac{d^2y}{dx^2} &= -0.00071 \end{aligned}$$

Substituting and evaluating

$$\begin{aligned} y_1 &= -0.00071 \\ y_2 &= -0.00071 \\ y_3 &= 0.00411 \end{aligned}$$

If we solve in the usual way

$$\begin{aligned} y^2 + 13x y + 12 = 0 \\ \frac{dy}{dx} = \frac{-13y}{4y^2 + 26x y} \\ \frac{dy}{dx} = \frac{(22y^2 - 670x y + 378x^2) \frac{dy}{dx}}{(4y^2 + 26x y)^2} \end{aligned}$$

when $x = 0$

$$y = \sqrt{3} \pm 1\sqrt{3} = 1.32 \pm 1.7321$$

$$\frac{dy}{dx} = -2.28 - 2.281$$

$$\frac{d^2y}{dx^2} = 162.9 \pm 0.1$$

We see that the real part of the series gives a divergent series and therefore the entire series is divergent.

McClintock Method of Enlargement.

In the "American Journal of Mathematics," 1895, Vol. XVII, Pages 90-130, Emory McClintock has solutions of and for a number of equations for all roots by this method. The theory of the method was written by him in a previous paper. The writer could not find a reference to this work. In any event a complete discussion of this method would not be included within the scope of this thesis, as a close study of the Journal shows that if the original papers could be found there would be enough material for a complete thesis in itself.

A knowledge of the theory is not necessary to solve equations. I did solve several equations and found the solutions to be very interesting, with the exception of the trinomial equation my opinion is the "Root Square Method" *is* the most practical.

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Iteration Method

This process may be applied to obtain a more exact complex root when an approximate root has been found. If in solving by MacLaurin's series or Mc Clintock's method a slowly converging series is found. This method may be used to determine a root which is more exact.

Given the function

$$g(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

If we have an approximate complex root e we may form the equation

$$m(x) \equiv [x - (a + bi)] [x - (a - bi)] = 0, \text{ where } e = a + bi,$$

since complex roots occur in pairs or

$$m(x) \equiv x^2 - 2ax + (a^2 + b^2) = 0.$$

then by the division algorithm

$$(1) \quad g(x) \equiv m(x) q(x) + r(x) \quad \text{where } r(x) \text{ is linear or zero}$$

now the substitution of either e or its conjugate in

(1) gives

$$g(x) \equiv r(x), \text{ then } r(e) \text{ gives the value of } g(e).$$

In like manner the value of $g'(x)$ may be evaluated for

$$x = e. \quad g'(x) \equiv m(x) q'(x) + r'(x) \quad g'(e) = r'(e)$$

then a more exact root f will be, $f = e - \frac{g(e)}{g'(e)}$,

Obviously this method may be repeated to give as exact a root as desired. In solving $x^4 + 7x - 6 = 0$ by McClintock's method the approximate complex roots are found to

Iteration Method

This process may be applied to obtain a more exact complex root when an approximate root has been found. It is solved by MacLaurin's series or the Olmstead's method a slowly converging series is found. This method may be used to determine a root which is more exact.

Given the function

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

 If we have an approximate complex root e we may form the

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$$m(x) \equiv [x - (e + bi)] [x - (e - bi)] = 0, \text{ where } e = a + bi$$

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 now the substitution of either e or its conjugate in

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 in like manner the value of $f'(e)$ may be evaluated for

$$x = e, \quad f'(x) \equiv m'(x)q'(x) + r'(x) \quad q'(e) = r'(e)$$

then a more exact root f will be, $f = e - \frac{f(e)}{f'(e)}$
 Obviously this method may be repeated to give as exact a root as desired. In solving $x^4 - 7x - 8 = 0$ by MacLaurin's method the approximate complex roots are found to

be $.6 \pm 1.7 i$

now form the equation

$$m(x) \equiv [x - (.6 \pm 1.7 i)] [x - (.6 - 1.7 i)] = 0.$$

$$m(x) \equiv x^2 - 1.2x + 2.89 = 0$$

now if $g(x) \equiv x^4 + 7x - 6 = 0$ then dividing $\frac{g(x)}{m(x)}$, and

$$\text{finding the remainder, } r(x) = 1.792x - 1.81$$

we have

$$g(e) = r(e) \text{ or}$$

$$g(.6 \pm 1.7 i) = 1.792x - 1.81$$

$$= -.735 \pm 3.046 i$$

$$\text{now } g'(x) \equiv 4x^3 + 7 = 0$$

$$\text{and } \frac{g'(x)}{m(x)} \text{ gives a remainder } r'(x) = -5.80x - 6.872$$

$$\text{then } g'(.6 \pm 1.7 i) = -5.80x - 6.872$$

$$= -10.352 - 9.86 i$$

now for a nearer approximation f

$$f = e - \frac{g(e)}{g'(e)}$$

$$f = (.6 \pm 1.7 i) - \frac{-.735 \pm 3.046 i}{-10.352 - 9.86 i}$$

$$f = (.6 \pm 1.7 i) - (.183 - .119 i)$$

$f = .417 \pm 1.819 i$ which is a more approximate value of the root.

The real roots were found to be $.800, -2.159$

$$\text{Check. } r_1 r_2 r_3 r_4 = -6.017$$

Check. $r_1^2 r_2^2 r_3^2 r_4^2 = -0.017$

The real roots were found to be .800, -2.152

of the roots.

$r = .417 \pm 1.619i$ which is a more approximate value

$$r = (.8 \pm 1.7i) - (.183 - .119i)$$

$$r = (.8 \pm 1.7i) - \frac{-.733 \pm 3.046i}{-10.338 - 9.881i}$$

$$r = e - \frac{f'(e)}{f''(e)}$$

now for a better approximation r

$$= -10.338 - 9.881i$$

$$\text{then } g'(.8 \pm 1.7i) = -5.80x - 6.873$$

$$\text{and } \frac{f'(x)}{f''(x)} \text{ gives a remainder } r'(x) = -5.80x - 6.873$$

$$\text{now } g'(x) = x^2 + 7 = 0$$

$$= -.733 \pm 3.046i$$

$$g(.8 \pm 1.7i) = 1.792x - 1.81$$

$$g(e) = r(e) \text{ or}$$

we have

$$\text{finding the remainder, } r(x) = 1.792x - 1.81$$

$$\text{now if } g(x) = x^2 + 7x - 8 = 0 \text{ then dividing } \frac{g(x)}{r(x)}, \text{ and}$$

$$m(x) = x^2 - 1.8x + 3.82 = 0$$

$$m(x) = [x - (.8 \pm 1.7i)][x - (.8 - 1.7i)] = 0.$$

now from the equation

$$be \quad .8 \pm 1.7i$$

SOLUTION of DE MOIVRES QUINTIC

Given the equation

$$(I) \quad x^5 - 5q x^3 + 5q^2 x - c = 0$$

Let $x = y + q/y$ then

$$(y + q/y)^5 - 5q (y + q/y)^3 + 5q^2 (y + q/y) - c = 0$$

expanding and combining terms gives

$$y^5 + \frac{q^5}{y^5} - c = 0$$

$$(y^5)^2 - c y^5 + q^5 = 0$$

Solving as a quadratic

$$(II) \quad y^5 = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - q^5}$$

$$\text{then } y_1 = \sqrt[5]{\frac{c}{2} + \sqrt{\frac{c^2}{4} - q^5}}, \quad y_2 = \sqrt[5]{\frac{c}{2} - \sqrt{\frac{c^2}{4} - q^5}}$$

then $y_1 y_2 = q$ and if e is a fifth root of unity the ten values of y can be separated into pairs $y_1 e^r$,

$y_2 e^{5-r}$ ($r = 0, 1, 2, \dots, 4$) such that the product of the

two in a pair is $y_1 y_2 = q$.

Now $x = y + q/y$ and the five roots of (I) are

$$(III) \quad x = y_1 e^r + y_2 e^{5-r} \quad (r = 0, 1, 2, \dots, 4)$$

Other forms of equations which can be solved by this method are

$$x^4 - 4q x^2 + 2q^2 x - c = 0$$

$$x^7 - 7q y^5 + 14 q^2 y^3 - 7q^3 y - c = 0$$

SOLUTION OF DE RIENS EQUATION

Given the equation

$$(1) \quad x^5 - 6x^4 + 3x^3 - 6x^2 + 3x - 6 = 0$$

Let $x = y + \sqrt{y}$ then

$$(y + \sqrt{y})^5 - 6(y + \sqrt{y})^4 + 3(y + \sqrt{y})^3 - 6(y + \sqrt{y})^2 + 3(y + \sqrt{y}) - 6 = 0$$

Expanding and combining terms gives

$$y^5 + \frac{5}{2}y^{\frac{9}{2}} - 6y^4 - 6y^{\frac{7}{2}} + 3y^3 + 3y^{\frac{5}{2}} - 6y^2 - 6y^{\frac{3}{2}} + 3y + 3y^{\frac{1}{2}} - 6 = 0$$

$$(y^{\frac{1}{2}})^{10} - 6(y^{\frac{1}{2}})^8 + 3(y^{\frac{1}{2}})^6 - 6(y^{\frac{1}{2}})^4 + 3(y^{\frac{1}{2}})^2 - 6 = 0$$

Solving as a quadratic

$$(11) \quad y^{\frac{1}{2}} = \frac{6 \pm \sqrt{36 - 6}}{2} = \frac{6 \pm \sqrt{30}}{2}$$

$$\text{Then } y = \left(\frac{6 \pm \sqrt{30}}{2} \right)^2 = \frac{36 \pm 12\sqrt{30} + 30}{4} = \frac{63 \pm 12\sqrt{30}}{4}$$

then $x_1 = y + \sqrt{y} = 4$ and 11 is a fifth root of unity. The ten

values of y can be separated into pairs y_1, y_2

$y_1, y_2 = 0, 1, 2, \dots, 10$ such that the product of the

two in a pair is $y_1 y_2 = 1$.

Now $x = y + \sqrt{y}$ and the five roots of (1) are

$$(12) \quad x = y_1^{\frac{1}{5}} + y_1^{\frac{2}{5}} \quad (r = 0, 1, 2, \dots, 4)$$

Other forms of equations which can be solved by this method

are

$$x^5 - 6x^4 + 3x^3 - 6x^2 + 3x - 6 = 0$$

$$x^5 - 6x^4 + 3x^3 - 6x^2 + 3x - 6 = 0$$

Example:

$$x^5 - 10x^3 + 20x - 31 = 0$$

$$-5q = 10, \quad 5q^2 = 20 - c = 31$$

$$q = 2, \quad q - \frac{1}{2}c = -31$$

$$y^5 = -\frac{31}{2} \pm \sqrt{\frac{961}{4} - (-32)}$$

$$y^5 = -\frac{31}{2} \pm \frac{33}{2}$$

$$y^5 = 1, -32$$

$$y = 1, -2$$

$$x = (e^r - 2e^{5-r}) \quad r = (0, 1, 2, \dots, 4)$$

where e is the fifth root of unity or

$$e^0 = 1, e^1 = .30902 + .95106i, e^2 = -.80902 + .58779i$$

$$e^3 = -.80902 - .58779i, e^4 = .30902 - .95106i$$

$$\text{when } r = 0 \quad x_0 = -1$$

$$r = 1 \quad x_1 = -.30902 + 2.85318i$$

$$r = 2 \quad x_2 = .80902 + 1.76337i$$

$$r = 3 \quad x_3 = .80902 - 1.76337i$$

$$r = 4 \quad x_4 = -.30902 - 2.85318i$$

$$\text{check } x_0 x_1 x_2 x_3 x_4 = -31.0005$$

Solution by MacLaurin's Series

If several of the terms of the equation were missing it might be advisable to try this method, and see if the series will converge quite rapidly. This can be done quite easily and the advantage would be if it should be a

Example:

$$x^5 + 10x^4 + 20x^3 + 31x^2 = 0$$

$$-31 = 10, \quad 20 = 20 - 10 = 10$$

$$10 = 10, \quad 2 = 2 - 10 = -8$$

$$y^5 = -\frac{31}{5} - \frac{10}{5} \sqrt{\frac{10}{5} - \frac{31}{5}}$$

$$y^5 = -\frac{31}{5} - \frac{10}{5} \sqrt{\frac{10}{5} - \frac{31}{5}}$$

$$y^5 = 1, \quad -31$$

$$y = 1, \quad -3$$

$$x = (e^{\frac{2\pi i}{5}} - 3e^{-\frac{2\pi i}{5}}) \quad r = (0, 1, 2, \dots, 4)$$

where e is the fifth root of unity or

$$e^0 = 1, \quad e^1 = .30902 + .98163i, \quad e^2 = -.30902 + .98163i, \quad e^3 = -.30902 - .98163i, \quad e^4 = .30902 - .98163i$$

$$e^0 = 1, \quad e^1 = .30902 + .98163i, \quad e^2 = -.30902 + .98163i, \quad e^3 = -.30902 - .98163i, \quad e^4 = .30902 - .98163i$$

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$$r = 3 \quad x_3 = .30902 - .98163i$$

$$r = 4 \quad x_4 = -.30902 - .98163i$$

$$\text{check } x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = -31.0000$$

SUMMARY

Root Square Method:

I believe this to be the best method if all the roots of the equation are to be determined. The only time difficulty might be encountered is when two or more of the absolute values of the real roots, or moduli of the complex roots, should happen to be quite near together. In this case the process might be very protracted and one of the other methods would be advisable.

The use of a table of squares will facilitate the work. Great care should be taken to make no mistakes in signs or products, since an error of this kind might easily lead to almost as much work as the original solution, and the solution is laborious enough in itself.

If several of the terms are missing in the equation then a solution by MacLaurin's Series or the method of McClintock might be advisable. But in this case a large number of the products in the Root Square Method would be zero for the first powers of the roots.

Solution by MacLaurin's Series

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SUMMARY

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If several of the terms are missing in the equation then a solution by MacLaurin's Series or the method of undetermined coefficients might be advisable. But in this case a large number of the products in the Root Square Method would be zero for the first powers of the roots.

Solution by MacLaurin's Series

If several of the terms of the equation are missing it might be advisable to try this method, and see if the series will converge rather rapidly. This can be done quite easily and the advantage gained if it should be a

rapidly converging series, will amply repay the work of the testing. In any other case my objection to this method is the work of the differentiations as they may become very involved. Also the calculation of the numerical values of these differentials.

The possibilities of linear transformations to make rapidly converging series should not be overlooked.

By this method the complex roots may be determined without regard to the real roots. In certain cases it might be advisable to do this and find the real roots by Horner's method, Newton-Raphson, or some other method if all roots are desired.

Iteration Method:

When under any method an approximate complex root has been found, the root may be found to any degree of accuracy by the repeated application of this method. If only complex roots are desired and the series under MacLaurin's Series are slowly convergent, then this method is very satisfactory. The proof that Newton's Method of approximation holds for complex roots as well as real roots may be found in "Theory of Equations--Todhunter." Pgs. 133-134. DeMoivre's Quintic and Analogous Forms:

If the equation can be put in one of these forms a quick and exact solution can be made. Especially if the various roots of one have been predetermined or are available.

General:

The theorems for the determination of real roots should be kept constantly in mind. In many cases it may be practical to use these to determine the real roots in conjunction with the methods for determining the complex roots.

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